Consistent Truncations of Supergravity through Generalised Geometry

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Abstract

Generalised geometry is a framework that unifies the diffeomorphisms, the gauge transformations and the fluxes of supergravity. The basic formalism of $O(d,d) \times R^+$ and $E_{7(7)} \times R^+$ generalised geometries is reviewed. It is then applied to find consistent truncations of ten- and eleven-dimensional supergravities. This formalism gives the structure of the truncated theory and includes different amounts of supersymmetry. In particular, the cases of half-maximal and quarter-maximal five-dimensional consistent truncations are considered. In the latter example, algebraic considerations from exceptional generalised geometry enable to greatly restrict the number of possible theories. To know whether these are realised, a further differential constraint must be solved. A specific construction of a $\mathcal{N} = 2$ consistent truncation of Type IIB supergravity retaining one hypermultiplet and two vector multiplets is conjectured but is shown not to work.

Acknowledgments

First and foremost, I want to thank Prof. Dan Waldram who so many times answered my questions in impromptu meetings, the lesson usually being: "actually things are not so simple". Even if this was sometimes said just after I thought I had finally grasped some concept - which is very slightly irritating! (but in a good way) - this really made working on this project the best part of QFFF. I would also like to thank Stephanie, who made herself available multiple times to answer my questions even when she was on the other side of the planet.

Tobi, the relentless patience with which you attacked everything, from our initial bafflement and confusions in generalised geometry to our exams really impressed me. I really hope you enjoy your new Californian life and PhD. Emma and Ray, you were my rocks during these two years, through the good and the bad, I'll never forget it (even if you still really need to improve at Cambio Ray).

When I started my masters, I never imagined that the hardest part of it would take the form it did. To my parents, my aunt and my sister, this year was difficult for all of us and I am unbelievably grateful to have you so closed, even if not in distance. Jojo, we all miss you incredibly. In what is hard to think was our last lunch, you asked me to summarise the very topic of this dissertation in three minutes. The result was confused and horrendous (hopefully I make it up now). But it doesn't matter. We laughed a lot.

To Jojo

Contents

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."
- Michael Atiyah [1]

1 Introduction

Throughout history, it is hard to say who from mathematicians or physicists stole more from the other. In part, this is because for the longest time these people were the same, but not only. The present case lies however in the second category.

Generalised geometry was initiated as generalised complex geometry in 2002 by Hitchin, who generalised the notion of Calabi-Yau manifolds as well as symplectic manifolds [2]. This unification was possible by taking seriously the idea that the complex and symplectic geometry can be thought on the sum $TM \oplus T^*M$, instead of as linear operations on them separately, which was taken up by Hitchin's students Gualtieri and Cavalcanti [3]. This construction gave rise to an operation (a bracket) which was already implicit in the work of Dorfman [4] although it was only presented in the present form by Courant [5]. It was then realised that this generalised geometry, later called $O(d,d)$ generalised geometry, provided a natural setting for for the NSNS sector of Type II supergravity [6], [7].

Supergravity first arose as the next logical step after the discovery of supersymmetry, itself being the natural way to unify the symmetries of the Standard Model by evading the Coleman-Mandula theorem [8] through the Haag-Łopuszański-Sohnius theorem [9].

The first example of supergravity was written in 1973 by Volkov and Soroka [10], but it became all the more relevant when it was discovered that it described the low-energy of string theory. With the advent of the second superstring revolution, the unique eleven-dimensional supergravity was seen as the low-energy limit of a theory unifies all five ten-dimensional string theories, called M-theory. This again prompted a better understanding of supergravity in these numbers of dimensions and their reductions, which reached its pinnacle when its language - generalised geometry - was discovered.

In spite of its mathematical origins, we are still doing physics and we will emphasize the physical origins rather than the more abstract story of algebroids, that start with Lie algebras and end - for now - with the so-called Y-algebroids, passing by Lie algebroids and Courant algebroids while still remaining within the realm of Leibniz algebroids [11]. The physical story on the other hand starts with the symmetries of the bosonic sector of ten- and eleven-dimensional supergravities and repackages them in a single geometric object (named the "generalised tangent bundle"). This is almost enough to see how the most basics elements of generalised geometry grow out of supergravity. The subtlety lies in the fact that contrary to electromagnetism, the analogues of the field strength are not necessarily two-forms. In supergravity, these analogues appear roughly as the bosonic content that is not the graviton and are named fluxes. This higher-form symmetry induces more structure on the generators in the form of their patching and leads to the definition of the generalised tangent bundle. From this, more bundles can be constructed such the generalised adjoint bundle.

The rest consists in finally generalising the conventional Riemannian geometry. Specifically, analogues of the Lie derivative, metric, connections and Ricci scalar among other things can be defined. It is obvious that the act of generalising a concept requires to drop some of the original's properties without an a priori clear choice between what is retained and what it is not and as with any generalisation, it is difficult to show precisely why one notion is enlarged in one way rather than another. Two things to do remain: first one can try to convey why a specific choice is at least reasonable, secondly we can show that it does lead to somewhere interesting. Fortuitously, the second part is easy in our case as in some important ways the geometry has already been generalised. The end result is that the bosonic part of Type II and eleven-dimensional supergravity can simply be written as a generalised Einstein-Hilbert action. Although no new physics has been found so far, it is clear for anyone familiar with the non-linearities of supergravity that the fact such a reformulation is at all possible is in itself remarkable.

At this stage, two important remarks should be made. First, while the basic bundles can be inferred from the original physical theory, it is not true that objects such as the generalised Lie derivative (and therefore the rest too) could be constructed. This is partly[1] why more emphasis was put on this first part of the construction rather than on the construction of the generalised Ricci and the comparison with the original supergravity action. The second point is that, although we only started by looking at the symmetries of the bosons, the fermions can automatically be incorporated by taking the double cover of the bosonic maximal compact subgroup as usual. Supersymmetry variations as well as fermionic equations of motions come out of the formalism in a natural manner. The reason for this can be broadly recognised as confirmation of the fact that what a physicist calls supersymmetry, a mathematician calls interesting geometry.

As these two remarks show, there is much motivation to view generalised geometry as the natural language of supergravity. And as usual, it is hoped that a new perspective, while elegant on its own, gives more than just that. Rightfully, expectations are raised to answer previously unsolved problems. For instance, so far, it has been used to understand the spectra of consistent truncations. Another use is in the description of supersymmetric backgrounds: generalised geometry enabled to calculate moduli of flux backgrounds that couldn't be calculated before, while holography is yet another application (like understanding the marginal deformations of supersymmetric backgrounds). We will only focus on the first of these uses here.

Even though it was not yet clear at the time, consistent truncations go back a century ago, when Kaluza supposed that the world was really five dimensional with only a metric as a field content. Assuming the "cylinder condition", he realised that this five-dimensional metric could be rephrased as a four-dimensional metric, along with a vector field (giving electromagnetism) and a scalar field [12]. In 1926, Klein gave a quantum and geometrical interpretations to Kaluza's idea by

[1] Along with the fact that generalised Lie derivatives and torsions are much more used in the second part. Also, generalised geometry is such a vast subject that we had to make a lot of choices in what we could cover.

suggesting that the cylinder condition originated from a small circular dimension [13]. Underlying their analysis was the fact that only the zeroth mode in the Fourier expansion of the metric was retained. This was in accord with the small radius of the fifth dimension and the fact that this dimension is unseen, meaning that the four dimensional theory is a low energy description of the universe. However, such type of truncation is in general not possible to do as we will explain in more detail later.

Ultimately, it became clear that the Kaluza-Klein theory could not describe our Universe. The main idea of dimensional reduction was nonetheless powerful enough to survive to this day through superstring theory. If one is interested in string phenomenology, the first obvious question is what manifolds should be used to compactify the extra dimensions. Contrarily to the seeming unicity of string theory coming from the second superstring revolution, this question unleashed a plethora of unfixed parameters. Today this is phrased in terms of the swampland (theories which do not admit a UV completion with gravity) and the string theory landscape (the collection of possible false vacua). Specifically, while waiting for some vacuum selection principle, it asks: how does the string theory landscape compare with the set of anomaly-free effective field theories?

Since the low-energy limit is given by supergravity, it is not so surprising that generalised geometry can help shed light on this deep question. In order to do that, one has to generalise the Kaluza-Klein theory to more general spaces than circles. Fluxless compactifications lead to Calabi-Yau manifolds or orbifolds. Generalised geometry can be used when fluxes are turned on (which is a wanted feature since it was discovered that they can generate warped metrics and break supersymmetry in a stable way). But even when this is done, not all questions have been answered. Indeed, a truncation scheme leading to the low-energy theory is still lacking. That only specific truncations are consistent can be seen in the fact that there is no reason to expect the following diagram (taken from [14]) to commute:

$$\begin{array}{ccc} \text{Full action } S & \longrightarrow & \text{Truncated action } S_t \\ \downarrow & & \downarrow \\ \text{Field equations } \delta S = 0 & \longrightarrow & \text{Truncated field equations } \delta S_t = 0. \end{array} \tag{1.1}$$

Consistent truncations are thus relevant in this context as they single out the supergravities that can be uplifted to string theory and M-theory from those that cannot.[2]

In [15] and later in [16], the most general known conditions for consistent truncations were found. Importantly, this finally included truncation ansätze for compactifications on sphere manifolds such as eleven dimensional supergravity on S^7 and S^4 which were known to be consistent, but eluded any attempt at a systematical understanding of the problem.

After briefly introducing some necessary mathematical tools and physical context[3], a summary of generalised geometry will be given. This summary will be aimed towards what will be needed when constructing the truncations in the second part. In particular, the emphasis will be less

[2]Note that the supergravities that are not retained are still perfectly consistent theories in their dimensions.

[3]This physical will be a brief presentation of some notions of gauged supergravity, but the examples of Type II and eleven-dimensional supergravities will only be considered in the beginning of the next chapter to show more closely its links with generalised geometry.

on the invariance of the Courant bracket, the Ricci tensor or the supersymmetry variations than on construction of the basic bundles, Dorfman derivative and general intrinsic torsion. This is both because of the length and because the former concepts are less immediately relevant to the construction of consistent truncations than the latter ones - although they are also relevant in the sense that they imply that generalised geometry captures the geometry of supergravity on which the central theorem of [16] rests.

To give a clearer and broader view of generalised geometry, each step in the construction of the theory - that is explained here - will be given in the context of the two types of generalised geometries, the simpler $O(d,d) \times R^+$ type and the more useful $E_{d(d)} \times R^+$ type. The first case can be used to describe the NSNS sector of Type II supergravity (and is the one more closely linked with the original general complex geometry of Hitchin) while the second case can be used for both Type II A/B and eleven-dimensional supergravity. We will only focus on the 11d case and give the corresponding formulas when needed later. Finally, note that these two group bear resemblance with the T and U duality groups of string theory respectively. However, the duality groups are defined over the integers while these groups are defined over the reals. Generalised geometry does not describe the geometry of the full string theory but is restricted to supergravity.

The consistent truncations part will start with a short explanation of what they are in supergravity before stating the central theorem of [16]. Since it is more than an existence theorem, we will apply it to two cases to show how it gives the structure of the truncated theory. In both cases we will be interested in Type IIB on a five-dimensional manifold, first preserving half-maximal then quarter-maximal, which was considered in [17] and [18]. These last two papers constrained the possible truncations and it remains to see individually whether these possibility are realised or not. For one such theory, a failed attempt at constructing the generalised adjoint tensors is finally presented.

2 Mathematical and Physical Preliminaries

2.1 G-structures

A principal bundle is a bundle with a fiber which is isomorphic to a group. The group action corresponds to a transformation of a vector from one neighborhood to another around the same point on the manifold. Furthermore the action of the group should be free and transitive on the fiber. Intuitively, this is to ensure that there always exists a neighborhood change corresponding to an element of the group.

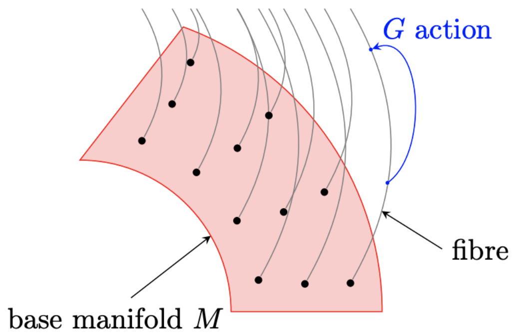

Figure 1: Representation of a principal fiber bundle (taken from [19])

An example of principal bundle is the frame bundle F where the group is $GL(d, \mathbb{R})$ (for an d-dimensional manifold) and is given by the set of all ordered bases of the elements of TM. A G-structure is a principal G-subbundle of the frame bundle.

For example, an $O(d)$-structure is equivalent to a metric structure [20]. First, we show how a metric can be obtained from that structure. Let $e_a \in T_xM$ and let $\hat{e}_a \in T_x^*M$ such that:

$$e_m^a \hat{e}_b^m = \delta_b^a. \tag{2.1}$$

The vielbeins define the components of the Riemannian metric as:

$$g_{mn} = \delta_{ab} e_m^a e_n^b. \tag{2.2}$$

Conversely, given a Riemannian metric g_{mn} at a point, we can always construct a set of orthonormal frames as:

$$P_x = \{\{\hat{e}_a\} \in F_x : g(\hat{e}_a, \hat{e}_b) = \delta_{ab}\}. \tag{2.3}$$

The metric is preserved for transformations of the type $\hat{e}'_a = M_a{}^b \hat{e}_b$, where $MM^T = \mathbb{I}$. Hence $M \in O(d)$.

The fact that a Riemannian metric at point is equivalent to an $O(d)$-structure means that we can alternatively define it as:

$$g_x \in \frac{GL(d, \mathbb{R})}{O(d, \mathbb{R})}. \tag{2.4}$$

The correspondence between a G-structure and globally defined invariant tensors is true in most cases. However, this usually carries some topological constraints on M, contrarily to the case of an $O(d)$-structure which can always be defined.

For instance, an $SL(d,\mathbb{R})$-structure is equivalent to a globally defined top form. This form can be thought of as the determinant of the frame. This carries a topological condition, namely the orientation (not all manifolds are orientable).

In general, smaller groups imply stronger restrictions. For a d-dimensional manifold, an $\mathbb{1}$-structure is equivalent to d linearly independent globally defined vector fields and the tangent bundle is a trival bundle. If a manifold admits an identity structure, it is said to be parallelisable. Lie groups are the archetypal examples of parallelisable manifolds (although parallelisable manifolds are not restricted to Lie groups). The only parallelisable spheres are S^0 (trivially), S^1 (since $U(1)$ is a Lie group), S^3 since ($SU(2)$ is a Lie group) and S^7 (which does not have a Lie group structure) [21].

A final example that will be useful later is the $Sp(d,\mathbb{R})$-structure, where d is even. This is also called an almost symplectic structure. This is equivalent to a globally defined non-degenerate two form Ω [20].

Given a subgroup A of B and C, the A-structure can usually be found by taking the globally invariant tensors equivalent to the B- and C-structures, along with some compatibility condition.

2.2 Exact Sequences and Semidirect Products

Let H and K be two groups and ϕ be a map from K to the automorphism group of H. The semidirect product $H \rtimes K$ is the set $H \times K$ endowed with the action:

$$(h,k)(h',k') = (h\phi_k(h'), kk'). \tag{2.5}$$

One can check the existence of the identity and inverse for any element as well as the associativity of this action. Hence, $H \rtimes K$ is a group [22].

Taking $A_1, ..., A_n$ to be spaces such as groups, vector spaces or modules, a sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} ... \xrightarrow{\phi_{n-1}} A_n \tag{2.6}$$

is exact if ker $\phi_i =$ im ϕ_{i-1} for $i \in \{2, ..., n-1\}$.

Consequently, we immediately have: $0 \to M \xrightarrow{\phi} N$ is exact if and only if ϕ is injective (since ker $\phi = 0$) and $M \xrightarrow{\phi} N \to 0$ is exact if and only if ϕ is surjective (since im $\phi = N$).

For an exact sequence $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P \to 0$, it is not in general possible to know what N is on the basis of M and N alone (one needs to know one of the maps as well). An exact sequence for which $N \cong M \oplus P$, with ϕ and ψ being the inclusion and projection maps respectively is called split. The splitting lemma states that a split exact sequence is equivalent to the existence of a homomorphism $\alpha : N \to M$ such that $\alpha \circ \phi = \mathrm{id}_M$, or a homomorphism $\beta : P \to N$ such that $\psi \circ \beta = \mathrm{id}_P$. If M, N and P are groups, the splitting lemma states that $N = M \rtimes P$. If M, N

and P are vector spaces, then the sequence is always split exact, i.e. $N \cong M \oplus P$. Finally, if M, N and P are groups and the sequence is again split exact, then $N = M \rtimes P$. [23]

Example: $G_{NS} = \Omega^2_{cl} \rtimes GL(d, \mathbb{R})$

Let $h \in \Omega^2_{cl}$ and $k \in GL(d, \mathbb{R})$ and consider the matrix multiplication:

$$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k^{-T} \end{pmatrix} = \begin{pmatrix} k & 0 \\ hk & k^{-T} \end{pmatrix} \tag{2.7}$$

We now have that:

$$\begin{pmatrix} k & 0 \\ hk & k^{-T} \end{pmatrix} \begin{pmatrix} k' & 0 \\ h'k' & k'^{-T} \end{pmatrix} = \begin{pmatrix} kk' & 0 \\ hkk' + k^{-T}h'k' & (kk')^{-T} \end{pmatrix} \tag{2.8}$$

Writing Eq. 2.8 in the form of Eq. 2.7 as:

$$\begin{pmatrix} kk' & 0 \\ hkk' + k^{-T}h'k' & (kk')^{-T} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h + k^{-T}h'k^{-1} & 1 \end{pmatrix} \begin{pmatrix} kk' & 0 \\ 0 & (kk')^{-T} \end{pmatrix}, \tag{2.9}$$

the action of Eq. 2.5 becomes apparent with $\phi_k(h') = k^{-T}h'k^{-1}$, which means that a semidirect product group element can be written in the form of Eq. 2.7.

Alternatively, from the above discussion, this group can equivalently be described by the split exact sequence:

$$1 \to \Omega^2_{cl} \to G_{NS} \to GL(d, \mathbb{R}) \to 1, \tag{2.10}$$

where 1 represents the identity group.

2.3 Elements of Gauged Supergravity

Supergravity is any theory of local supersymmetry. A rigid symmetry is made local by introducing a gauge field. For instance, a $U(1)$ symmetry parameter α is made local by defining a $U(1)$ gauge field A_μ such that $\delta A_\mu = \partial_\mu \alpha$. In supersymmetry, the transformation paramater ϵ_α is fermionic. Hence the gauge field $\psi_{\mu\alpha}$ contains a spin-1 (the vector μ index) part and a spin-1/2 (the spinor α index) part, which constrain ψ to be a spin-3/2 field (called the gravitino) whose supersymmetric partner is a spin-2 field (called the graviton). Supergravity is hence equivalently defined as a supersymmetric theory of gravity.

This equivalence between local supersymmetry and supersymmetry with gravity is one reason to study supergravity. Another motive for its study is the fact that ten-dimensional supergravities are low energy limits of the non-pertubatively related versions of string theory, while eleven-dimensional supergravity is the low-energy of M-theory.

The simplest manifold on which type II or eleven-dimensional supergravity can be compactified is a torus T^n. The resulting theory possesses maximal supersymmetry as well as a global and an abelian local symmetry. However, the matter content is not charged under this abelian local symmetry. These theories are therefore called ungauged supergravities. In ungauged supergravity, the scalar fields transform in a non-linear representation of some global group G. Explicitely, this means that the Lagrangian is invariant under the transformation of the matrix $\mathcal{V}$ of scalar [24]:

$$\delta\mathcal{V} = \Lambda\mathcal{V} - \mathcal{V}k(x), \tag{2.11}$$

where $\Lambda = \Lambda^\alpha t_\alpha \in \text{Lie } G$ and $k(x) \in \text{Lie } H$, where H is the maximal compact subgroup of G.[4] This means that the scalars parametrise the coset $\frac{G}{H}$. For instance, for eleven-dimensional compactified on T^7, $G = E_{7(7)}$ and $H = SU(8)$. One sometimes fixes the local symmetry to a specific gauge such as the unitary gauge or the triangular gauge, although we will not do that here.

The Lagragian will also be invariant under the n_V vector fields with global and local transformations:

$$\delta A_\mu^M = -\Lambda^\alpha (t_\alpha)_N{}^M A_\mu^N,\ \delta A_\mu^M = \partial_\mu \Lambda^M \tag{2.12}$$

where the t_α are in the fundamental representation of Lie G, $1 \leq M, N \leq n_V$ and where we see that the abelian local symmetry mentioned above is $U(1)^{n_V}$. In general p-forms transform similarly in some specific representation of Lie(G) and accompanied by some tensor local symmetry.

As usual one can gauge the theory by making the replacement:

$$\partial_\mu \to D_\mu = \partial_\mu - gA_\mu^M \Theta_M{}^\alpha t_\alpha, \tag{2.13}$$

using the n_V vector fields A_μ^M and where $\Theta_M{}^\alpha t_\alpha \in \text{Lie } G$. Θ is a constant tensor called the embedding tensor. The indices α and M denote respectively the adjoint and fundamental representations of G. In general the rank of the embedding will not be maximal and correspond the dimension of the gauge group $G_0 \subset G$. Therefore, the embedding tensor can be viewed as a map:

$$\Theta : V \to \text{Lie } G, \tag{2.14}$$

where V is some vector space and where im $V = \text{Lie } G_{\text{gauge}}$.

The embedding tensor satisfies two consistency requirements. The first is a linear constraint coming from supersymmetry which reads:

$$\mathbf{P}\Theta = 0, \tag{2.15}$$

where $\mathbf{P}$ is some projector that restricts that the representations appearing in the tensor product of the fundamental and adjoint representations in $\Theta_M{}^\alpha$. The second consistency requirement is the quadratic constraint:

$$[X_M, X_N] = -X_{MN}{}^P X_P,\ X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P. \tag{2.16}$$

It can be understood as the requirement that the gauge algebra closes. It can equivalently be

[4] Note that there is no propagating gauge fields associated with this local symmetry.

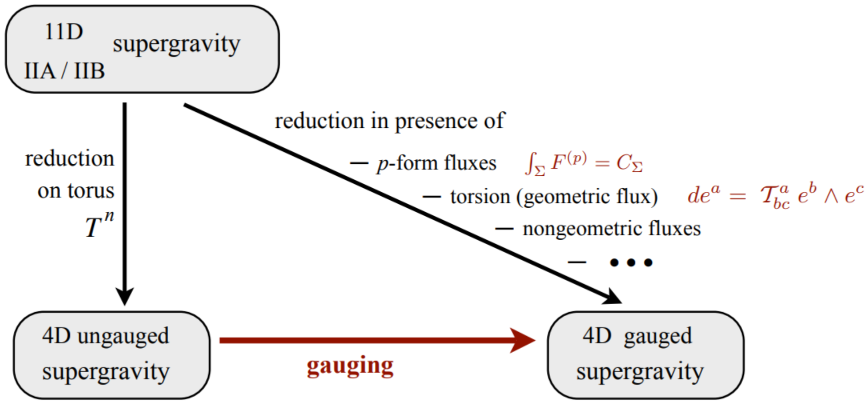

Figure 2: Ways to obtain gauged supergravity. Image taken from [24].

rewritten as:

$$\mathbf{P}'(\Theta \otimes \Theta) = 0, \tag{2.17}$$

for again some projector $\mathbf{P}'$ that picks out representations among the possible ones in the tensor product.

No higher order constraint is necessary as it turns out that imposing these two conditions is sufficient [25].

Intead of compactifying on a torus and taking a subgroup of the global symmetry group, gauged supergravity can also be obtained by directly compactifying on more complicated manifolds, with or without fluxes, as shown in Fig. 2.

3 Generalised Geometry

3.1 Generalised Tangent Bundle

The first aim of generalised geometry is to unify the different fields (including fluxes) of some supergravity. Before that, generalised geometry unifies the bosonic symmetries in a generalised tangent bundle. Here we illustrate what this means by constructing the generalised tangent bundles relevant for two cases: the NSNS sector of Type II supergravity (based on the $O(d,d)\times\mathbb{R}$ structure group) and 11 dimensional supergravity in the specific case of a 7 dimensional reduction (based on the $E_{7(7)}\times R^+$ structure group). However it should be noted that generalised geometry is powerful enough to also describe 11 dimensional supergravity compactifications ($d\leq 7$), and the full type IIA and type IIB supergravities. The main sources for this section is [26] for the $O(d,d)\times R^+$ case and [27] and [28] for the $E_{7(7)}\times R^+$ case.

3.1.1 $O(d,d)\times\mathbb{R}^+$ Generalised Geometry

The bosonic[5] part of Type II supergravity is defined by the pseudo-action[6] [26]:

$$S_B = \frac{1}{2\kappa^2}\int_{M_{9,1}}\sqrt{-g}[e^{-2\phi}(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H^2)-\frac{1}{4}\sum_n\frac{1}{n!}(F^{(B)}_{(n)})^2]. \tag{3.1}$$

The first term describes the NSNS (after Neveu and Schwartz) sector while the second one is the RR (after Ramond) part. The RR field obeys the self-duality relation:

$$F^{(B)}_{(n)} = (-1)^{\lfloor n/2\rfloor} * F^{(B)}_{(10-n)}, \tag{3.2}$$

where g is a ten dimensional metric, $\mathcal{R}$ is the Ricci scalar formed from the metric, ϕ is a scalar called the dilaton. Setting the fermions to 0, the action gives that:

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = \frac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho}+2\nabla_\mu\nabla_\nu\phi-\frac{1}{4}e^{2\phi}\sum_n\frac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1..\lambda_{n-1}}F_{\nu}^{(B)\lambda_1\ldots\lambda_{n-1}} &= 0,\\
\nabla^\mu(e^{-2\phi}H_{\mu\nu\lambda})-\frac{1}{2}\sum_n\frac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1..\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} &= 0,\\
\nabla^2\phi-(\nabla\phi)^2+\frac{1}{4}\mathcal{R}-\frac{1}{48}H^2 &= 0,\\
\mathrm{d}F^{(B)}-H\wedge F^{(B)} &= 0,\\
\mathrm{d}H &= 0,
\end{aligned} \tag{3.3}$$

[5]Because it is not immediately relevant for our purposes, the fermionic part (given by a pair of chiral gravitini $\psi^{\pm}_{\mu}$ and a pair of chiral dilatini $\lambda^{\pm}$) is ignored here.
[6]It is only a "pseudo-action" because Eq. 3.2 is a separate condition that does not follow from Eq. 3.1.

which is consistent with the "A-basis" where $A^{(n)}$ are n[7]-form potentials such that:

$$F^{(B)} \equiv \sum_n F^{(B)}_{(n)} = \sum_n e^B \wedge dA_{(n-1)}, \tag{3.4}$$

$$H = \mathrm{d}B_i \quad \text{on a patch } U_i, \tag{3.5}$$

with B a local two-form, $e^B = 1 + B + \frac{1}{2}B\wedge B + \frac{1}{3!}B\wedge B\wedge B + ...$

The bosonic content of the theory having been specified, we turn to their symmetries. First, because of the metric, this theory possesses the usual diffeomorphism invariance through a vector v. Secondly, H enjoys a type of gauge invariance. However, it is unusual as H is a three-form in contrast to the prototypical example of electromagnetism where the invariant object is the two-form field strength. The present case is more akin to a gauge transformation of a gauge transformation as the "gauge" field B itself enjoys gauge transformations. [29; 30; 31; 32] First, because H is globally defined, we have from Eq. 3.5:

$$\mathrm{d}(B_{(i)} - B_{(j)}) = 0 \Rightarrow B_{(i)} - B_{(j)} = \mathrm{d}\Lambda_{ij}. \tag{3.6}$$

on $U_i \cap U_j$. Similarly, plugging Eq. 3.4 in the penultimate line of Eq. 3.3, we obtain:

$$A_{(i)} = e^{\mathrm{d}\Lambda_{(ij)}} \wedge A_{(j)} - \mathrm{d}\hat{\Lambda}_{(ij)}. \tag{3.7}$$

where $\hat{\Lambda}_{(ij)}$ is a polyform (that go up to rank 8) and $\Lambda_{(ij)}$ is a one-form.

Since:

$$B_{(i)} - B_{(j)} + B_{(j)} - B_{(k)} + B_{(k)} - B_{(i)} = 0 \Rightarrow \mathrm{d}(\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)}) = 0. \tag{3.8}$$

On $U_i \cap U_j \cap U_k$, $\Lambda_{(ij)}$ must therefore obey the consistency relation:

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)}.^{8} \tag{3.9}$$

Mathematically, this relation means that B is called a connective structure on a gerbe. [33; 34; 35]

Eq. 3.6-3.7, summarise all the bosonic symmetries of Type II supergravity, however we are only interested in the NSNS symmetries, which means we can ignore Eq. 3.7. The reason we could not ignore $F^{(B)}$ from the start was because it depends on B which means that the NSNS symmetries act on it and we needed to make sure that the RR sector did not impact the NSNS symmetries, i.e. $\hat{\Lambda}_{(ij)}$ does not appear in Eq. 3.6 (in fact, the opposite happens since $\Lambda_{(ij)}$ appears in Eq. 3.7, which is fine).

Re-expressing Eq. 3.6 equivalently on the same patch[9] - the more common approach to gauge

[7] n is even for type IIB and odd for type IIA.
[8] If H is quantised, further conditions which we skip are necessary.
[9] This is done by replacing the form on $U_{(j)}$ by a primed form on $U_{(j)}$ and replacing $d\Lambda_{(ij)}$ by $d\lambda_{(i)}$.

symmetry taught in physics - we have[10]:

$$B'_{(i)} = B_{(i)} - d\lambda_{(i)}. \tag{3.10}$$

This implies:

$$\delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}. \tag{3.11}$$

Since the gauge symmetry does not interact with the metric and the dilaton, they enjoy the usual diffeomorphism invariance (generated by $\mathcal{L}_v$), infinitesimally given by:

$$\begin{aligned} \delta_{v+\lambda} g &= \mathcal{L}_v g, \\ \delta_{v+\lambda} \phi &= \mathcal{L}_v \phi. \end{aligned} \tag{3.12}$$

While we have by now reformulated the NSNS symmetries in different ways, our aim is to express all the symmetries on the same footing, which means in terms of v_i and λ_i directly (not $\mathrm{d}\lambda_{(i)}$). Note that $\mathrm{d}\Lambda_{(ij)}$ defines the patching for any form in the sense that:

$$\begin{aligned} B_i &= B_j + \mathrm{d}\Lambda_{(ij)}, \\ B'_i &= B'_j + \mathrm{d}\Lambda_{(ij)}, \end{aligned} \tag{3.13}$$

where $B'_i = B_i + \delta B_i$ and $B'_j = B_j + \delta B_j$. This implies that $\delta B_i = \delta B_j = \delta B$.

The patching of λ_i can then be found as follows:

$$\begin{aligned} \delta B = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)} &= \mathcal{L}_v B_{(j)} - \mathrm{d}\lambda_{(j)} \\ \Rightarrow \mathrm{d}\lambda_{(i)} &= \mathrm{d}\lambda_{(j)} + \mathcal{L}_v (B_{(i)} - B_{(j)}) \\ &= \mathrm{d}\lambda_{(j)} + \mathcal{L}_v \mathrm{d}\Lambda_{(ij)} \\ &= \mathrm{d}\lambda_{(j)} + \mathrm{d}(i_v \mathrm{d}\Lambda_{(ij)}) \\ \Rightarrow \lambda_{(i)} - (\lambda_{(j)} + i_v d\Lambda_{(ij)}) &= \mathrm{d}f, \end{aligned} \tag{3.14}$$

where $\mathrm{d}f$ is an integration constant that we now choose to be 0.

In summary, with this choice, we have:

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + (\lambda_{(j)} + i_v d\Lambda_{(ij)}). \tag{3.15}$$

Since the symmetries of the NSNS sector of type II supergravity are captured by a vector and a one-form, one might guess that the object of this generalised symmetry is simply an element of:

$$E = TM \oplus T^*M. \tag{3.16}$$

This is almost true, but it needs to be generalised to account for Eq. 3.15. This can be done by seeing that the transformation of Eq. 3.15 is obtained by acting a matrix of the form of Eq. 2.8

[10]For completeness, the NSNS symmetry on A would become $A'_{(i)} = e^{d\lambda_{(i)}} A_{(i)}$

on a vector of the form $\begin{pmatrix} v_{(i)} \\ \lambda_{(i)} \end{pmatrix}$.[11] This means that the structure group of the NSNS sector is none other than $G_{NS} = \Omega^2_{cl} \rtimes GL(d,\mathbb{R})$, which, as we saw, can be written as the split exact sequence of Eq. 2.10. The closed 2-form acts on T^*M, which means that the vector space E corresponding to G_{NS} is given by the split exact sequence:

$$0 \to T^*M \to E \to TM \to 0. \tag{3.17}$$

As a check, we can see that E is in fact isomorphic to the naive guess of Eq. 3.16 (using the discussion on split exact sequences on vector spaces), but now is general enough to include the twist present on the one-form.

Writing formally $V = v + \lambda \in E$ the generalised tangent bundle E can now be endowed with a natural metric given by:

$$\langle V, V \rangle = i_v \lambda = v^\mu \lambda_\mu = V^A \eta_{AB} V^B, \quad \eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix} \tag{3.18}$$

using capitalised letters for the generalised geometry notation (i.e. $V^A = v^\mu$ for $1 \leq A, \mu \leq d$ and $V^A = \lambda_\mu$ for $d+1 \leq A \leq 2d$). This means that an $O(d,d)$-principal bundle is naturally constructed, which is seen in the so-called conformal basis $\{\hat{E}_A\} = \{\frac{\partial}{\partial x^\mu}\} \cup \{\mathrm{d}x^\mu\}$, where :

$$\langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB}, \tag{3.19}$$

where:

$$O(d,d) = \{M \in GL(2d,\mathbb{R}) | (M^{-1})^C{}_A (M^{-1})^D{}_B \eta_{CD} = \eta_{AB}\}. \tag{3.20}$$

This natural metric is preserved under a twist of the form of Eq. 3.15. In a way, this can be seen as justifying the choice of Eq. 3.15 or rather the patching was chosen so as to be to compatible with an $O(d,d)$-structure. The terminology of generalised vector for $V = V^A \hat{E}_A$ and generalised tangent bundle for E can finally be justified by the noticing the following usual transformations of a vector's bases and components, only with respect to $O(d,d)$ here:

$$\hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A, \;\; V^A \mapsto V'^A = M^A{}_B V^B, \;\; M \in O(d,d). \tag{3.21}$$

Now that the symmetries have been unified in a generalised tangent bundle, the next step is to unify the actual degrees of freedom such as the metric g. This will be done in Section 3.3. There is however an exception as one of the degrees of freedom, ϕ, must be included at this stage already in order to correctly describe the NSNS sector of Type II supergravity, even though it does not generate any symmetry the way v or λ do.

This is done by weighting E by a real number (1 degree of freedom is needed) chosen to be det

[11] The action of a form on a vector is naturally given by the interior product.

T^*M such that:

$$\tilde{E} = \det T^*M \otimes E, \tag{3.22}$$

which implies that the principal bundle in fact has fibre $O(d,d) \times \mathbb{R}^+$ [12] as seen again with the conformal basis which now satisfies:

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB}, \quad \Phi \in \Gamma(\det\, (T^*M)), \tag{3.23}$$

such that Eq. 3.21 is still true, except $M \in O(d,d) \times R^+$.

The next useful bundle to consider is the adjoint bundle. Given the definition of $O(d,d)$ from Eq. 3.20, its Lie algebra is:

$$\mathfrak{o}(d,d) = \{x \in \mathfrak{gl}(V) \mid \eta(xV,W) + \eta(V,xW) = 0 \quad \forall V,W\}. \tag{3.24}$$

This is solved by [19]:

$$x = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}, \tag{3.25}$$

such that $A \in End(TM), \beta \in \Lambda^2 TM, B \in \Lambda^2 T^*M$. Taking into account the R^+ factor, this means that the adjoint bundle has the $GL(d,\mathbb{R})$ decomposition:

$$\text{ad}\tilde{F} \cong \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M, \tag{3.26}$$

which is indeed $\frac{2d(2d-1)}{2} + 1$ dimensional.

Because of this $GL(d,\mathbb{R})$ decomposition, the natural $\mathfrak{gl}(d,\mathbb{R})$ action (given for example on a vector v and a three-form λ):

$$(r\cdot v)^a = r^a{}_b v^b, \quad (r\cdot\lambda)_{abc} = -r^d{}_a \lambda_{dbc} - r^d{}_a \lambda_{adc} - r^d{}_a \lambda_{abd} \tag{3.27}$$

can be used to define the adjoint action $R \cdot V$, where $R \in \Gamma(\text{ad}\tilde{F})$. Indeed, applying this action in our context gives:

$$\left.\begin{array}{c} (\beta\cdot\lambda)^a = -\beta^{ab}\lambda_b \implies \beta\cdot(\beta\cdot\lambda) = 0 \\ (\beta\cdot v) = 0 \end{array}\right\} \implies \beta\cdot(\beta\cdot V) = 0. \tag{3.28}$$

Consequently, β is nilpotent of degree two[13], which renders the group action very easy:

$$e^\beta \cdot V = (1+\beta)\cdot V = (v - \beta \lrcorner \lambda) + \lambda. \tag{3.29}$$

Using the same reasoning, the action of B is also found to be nilpotent and Eq. 3.21 can be rewritten as $M \cdot V = e^c e^\beta e^B m \cdot V$, where e^c gives the R^+ scaling and m is the standard $GL(d,\mathbb{R})$ action.

[12] $O(d,d) \times \mathbb{R}^+ = \{M \in GL(2d,\mathbb{R}) | (M^{-1})^C{}_A (M^{-1})^D{}_B \eta_{CD} = \sigma^2 \eta_{AB}\}$.
[13] β is trivially nilpotent when acting on v.

3.1.2 $E_{d(d)} \times R^+$ Generalised Geometry

The bosonic part[14] of the 11 dimensional supergravity action is [28]:

$$S_{11,B} = \frac{1}{2\kappa^2}\int_{M_{10,1}} \mathrm{vol}_g \mathcal{R} - \frac{1}{2}\mathcal{F}\wedge *\mathcal{F} - \frac{1}{6}\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}, \tag{3.30}$$

where the only bosonic fields are the metric $g_{\mu\nu}$ and a three-form $\mathcal{A}_{\mu\nu\rho}$, such that $\mathcal{F} = \mathrm{d}\mathcal{A}$ and $\mathcal{R}$ is the Ricci scalar. The equations of motion and the Bianchi identity are then [36]:

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{1}{12}(\mathcal{F}_{\mu\rho_1\rho_2\rho_3}\mathcal{F}_\nu{}^{\rho_1\rho_2\rho_3} - \frac{1}{12}g_{\mu\nu}\mathcal{F}^2) = 0, \\ \mathrm{d} * \mathcal{F} + \frac{1}{2}\mathcal{F}\wedge\mathcal{F} = 0, \\ \mathrm{d}\mathcal{F} = 0 \end{aligned}$$

We now arrive at a point where a first subtlety compared to previous case must be pointed out. Here, the dimensional reduction must be made before finding the bosonic symmetries and constructing the generalised tangent bundle. This is in contrast with the $O(d,d)$ case where there was no mention of any splitting of the ten-dimensional spacetime.

So, as already mentioned, we will focus on compactifying eleven-dimensional supergravity on a seven-dimensional compact manifold M_7 such that[15]:

$$M_{10,1} = M_{3,1} \times M_7. \tag{3.31}$$

This implies that g and $\mathcal{F}$ decompose under $Spin(3,1) \times Spin(7)$ as:

$$\begin{aligned} \mathrm{d}s^2 = (\det g^{(7)})^{-1/2} g^{(4)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g^{(7)}_{mn}\mathrm{d}x^m\mathrm{d}x^n, \\ \mathcal{F} = F + *_7\tilde{F}\wedge(\det g^{(7)})^{-1} vol_{g^{(4)}}, \end{aligned} \tag{3.32}$$

where F and $\tilde{F}$ are four- and seven- forms respectively on the seven dimensional manifold (as we keep only the $Spin(3,1)$ scalars). Plugging in Eq. 3.32 into Eq. 3.1.2, one obtains:

$$\mathrm{d}F = 0, \tag{3.33}$$

$$\mathrm{d}\tilde{F} + \frac{1}{2}F\wedge F = 0, \tag{3.34}$$

$$\mathrm{d}((\det g^{(7)})^{-1} *_7 \tilde{F}) = 0, \tag{3.35}$$

$$\mathrm{d}((\det g^{(7)})^{-1} *_7 F) + (\det g^{(7)})^{-1} *_7 \tilde{F}\wedge F = 0, \tag{3.36}$$

Similarly to the previous case, we wish to unify the symmetries by finding the “gauge fields of the gauge fields” and choosing a twist compatible both with the patching obtained from the

[14] We again ignore the fermionic part (given by the gravitino ψ_μ only).

[15] In fact, only the tangent space to $M_{10,1}$ needs to be decomposable into a four-dimensional and a seven-dimensional parts, which is weaker than what we consider here.

supergravity and the structure group of the bundle we will obtain (which will be $E_{7(7)}$ here).

Before anything else, we find the first "level" of gauge fields (equivalent of the local A[16] and B and their patching (equivalent of Eq. 3.10). By Poincaré's lemma, Eq.3.33 implies that on a patch $U_{(i)}$:

$$F = \mathrm{d}A_{(i)}, \tag{3.37}$$

where $A_{(i)} \in \bigwedge^3 T^*M_7$. Now, up to an locally exact part which we will call $\mathrm{d}\tilde{A}_{(i)}$ (with $\tilde{A}_{(i)} \in \bigwedge^6 T^*M_7$) we can find $\tilde{F}$ by integrating Eq. 3.34:

$$\int \mathrm{d}\tilde{F} = -\frac{1}{2}\int F \wedge \tilde{F} = -\frac{1}{2}\int \mathrm{d}A_{(i)} \wedge F = -\frac{1}{2}\int \mathrm{d}(A_{(i)} \wedge F), \tag{3.38}$$

using Eq. 3.37 in the last step. Hence, on $U_{(i)}$:

$$\tilde{F} = \mathrm{d}\tilde{A}_{(i)} - \frac{1}{2}A_{(i)} \wedge F. \tag{3.39}$$

Plugging this back into Eq. 3.35, 3.36, we find that this solution is consistent. The next step is to deduce the patching, which is immediate. Since F is a globally defined form and is equal to itself while A is only defined locally, Eq. 3.37 implies on $U_{(i)} \cap U_{(j)}$:

$$\mathrm{d}A_{(i)} = \mathrm{d}A_{(j)}. \tag{3.40}$$

Using the same reasoning on Eq. 3.39, we have:

$$\mathrm{d}\tilde{A}_{(i)} - \frac{1}{2}A_{(i)} \wedge F = \mathrm{d}\tilde{A}_{(j)} - \frac{1}{2}A_{(j)} \wedge F, \tag{3.41}$$

which, using Eq. 3.37, is equivalent to:

$$\mathrm{d}\tilde{A}_{(i)} - \mathrm{d}\tilde{A}_{(j)} = \frac{1}{2}(A_{(i)} - A_{(j)}) \wedge \mathrm{d}A_{(i)}. \tag{3.42}$$

Eq. 3.40 and 3.42 specify the patching of $A_{(i)}$ and $\tilde{A}_{(i)}$, however they are currently written in terms of them as well as their exterior derivatives, which can be simplified. Using again Poincaré's lemma, Eq. 3.40 can be written as:

$$\mathrm{d}(A_{(i)} - A_{(j)}) = 0 \Rightarrow A_{(i)} - A_{(j)} = \mathrm{d}\Lambda_{(ij)}, \tag{3.43}$$

[16]Following convention, we use A for the gauge field of both the NSNS sector of type II and of eleven-dimensional supergravities so this A should not to be confused with that of Eq. 3.37 for instance.

for some $\Lambda_{(ij)} \in \bigwedge^2 T^*M_7$. Similarly, Eq. 3.42 can be simplified to:

$$\begin{aligned}
\mathrm{d}(\tilde{A}_{(i)} - \tilde{A}_{(j)}) = \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}A_{(i)} = -\frac{1}{2}\mathrm{d}(\mathrm{d}\Lambda_{(ij)} \wedge A_{(i)}) \\
\Rightarrow \mathrm{d}(\tilde{A}_{(i)} - \tilde{A}_{(j)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(i)}) = 0 \\
\Rightarrow \tilde{A}_{(i)} - \tilde{A}_{(j)} = \mathrm{d}\tilde{\Lambda}_{(ij)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(i)},
\end{aligned} \tag{3.44}$$

for some $\tilde{\Lambda}_{(ij)} \in \bigwedge^5 T^*M_7$. Following the $O(d,d)$ case, this would mean that the generalised tangent bundle E is composed of a global vector (generating the diffeomorphism) and the "gauge of the gauge" field, i.e. a two-form and a five-form. Knowing by now the bosonic symmetries, we would like to find their patching as before. We apply the same reasoning. Eq. 3.43 has the same form as Eq. 3.6 so we have[17]:

$$\delta A = \mathcal{L}_v A_{(i)} - \mathrm{d}\omega_{(i)} = \mathcal{L}_v A_{(j)} - \mathrm{d}\omega_{(j)} \tag{3.45}$$

Choosing the same integration constant leads to again:

$$\omega_{(i)} = \omega_{(j)} + i_v \mathrm{d}\Lambda_{(ij)}. \tag{3.46}$$

In addition, the vector generating the diffeomorphism is again global so that:

$$v_{(i)} = v_{(j)}. \tag{3.47}$$

To find the patching of $\sigma_{(i)}$, we first note that repeating the procedure of Eq. 3.13 leads to $\delta\tilde{A}_{(i)} = \delta\tilde{A}_{(j)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)}$.

[17] $\lambda_{(i)}$ is changed to $\omega_{(i)}$.

Repeating the procedure for Eq. 3.44 gives:

$$\begin{aligned}
\tilde{A}'_{(i)} &= \tilde{A}_{(i)} - \mathrm{d}\sigma_{(i)} + \frac{1}{2}\mathrm{d}\omega_{(i)} \wedge A_{(i)} \\
\Rightarrow \delta\tilde{A}_{(i)} = \mathcal{L}_v\tilde{A}_{(i)} - \mathrm{d}\sigma_{(i)} + \frac{1}{2}\mathrm{d}\omega_{(i)} \wedge A_{(i)} &= \mathcal{L}_v\tilde{A}_{(j)} - \mathrm{d}\sigma_{(j)} + \frac{1}{2}\mathrm{d}\omega_{(j)} \wedge A_{(j)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
\Rightarrow \mathrm{d}\sigma_{(i)} = \mathrm{d}\sigma_{(j)} + \mathcal{L}_v(\tilde{A}_{(i)} - \tilde{A}_{(j)}) &+ \frac{1}{2}(\mathrm{d}\omega_{(i)} \wedge A_{(i)} - \mathrm{d}\omega_{(j)} \wedge A_{(j)}) + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
= \mathrm{d}\sigma_{(j)} + \mathcal{L}_v(\mathrm{d}\tilde{\Lambda}_{(ij)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(i)}) &+ \frac{1}{2}(\mathrm{d}\omega_i \wedge A_{(i)} - \mathrm{d}\omega_{(j)} \wedge A_{(j)}) + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
= \mathrm{d}\sigma_{(j)} + \mathcal{L}_v\mathrm{d}\tilde{\Lambda}_{(ij)} - \frac{1}{2}(\mathcal{L}_v\mathrm{d}\Lambda_{(ij)} \wedge A_{(i)} + \mathrm{d}\Lambda_{(ij)} \wedge \mathcal{L}_v A_{(i)}) &+ \frac{1}{2}(\mathcal{L}_v(A_{(i)} - A_{(j)}) + \mathrm{d}\omega_{(j)}) \wedge A_{(i)} \\
&- \frac{1}{2}\mathrm{d}\omega_{(j)} \wedge A_{(j)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
= \mathrm{d}\sigma_{(j)} + \mathcal{L}_v\mathrm{d}\tilde{\Lambda}_{(ij)} + \frac{1}{2}\mathcal{L}_v(\mathrm{d}\Lambda_{(ij)} + A_j) \wedge \mathrm{d}\Lambda_{(ij)} &- \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\omega_{(j)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
= \mathrm{d}\sigma_{(j)} + \mathcal{L}_v\mathrm{d}\tilde{\Lambda}_{(ij)} + \frac{1}{2}(\mathcal{L}_v\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} &- \mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\omega_{(j)} + \mathcal{L}_v A_{(j)} \wedge \mathrm{d}\Lambda_{(ij)} \\
&- \mathrm{d}\Lambda_{(ij)} \wedge \mathcal{L}_v A_{(j)}) + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} \\
= \mathrm{d}\sigma_{(j)} + \mathcal{L}_v\mathrm{d}\tilde{\Lambda}_{(ij)} + \frac{1}{2}\mathcal{L}_v\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} &- \mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\omega_{(j)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge \delta A_{(i)}
\end{aligned} \tag{3.48}$$

Integrating (with a choice of integration constant of 0) gives:

$$\sigma_{(i)} = \sigma_{(j)} + \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge i_v\mathrm{d}\Lambda_{(ij)} + i_v\mathrm{d}\tilde{\Lambda}_{(ij)}. \tag{3.49}$$

It would seem that the generalised tangent bundle is composed of v, $\omega_{(i)}$ and $\sigma_{(i)}$, with the patching specified by Eq. 3.47, 3.46 and 3.49. However, a second subtlety occurs again here. We know from Section 2.3 that we are looking to fit these degrees of freedom in the fundamental **56** representation of $E_{7(7)}$, meaning that we are missing $56 - 7 - 21 - 21 = 7$ degrees of freedom.

These missing degrees of freedom are found by taking $GL(7, \mathbb{R}) \subset SL(8, \mathbb{R}) \subset E_{7(7)}$:

$$E_0 \cong TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M), \tag{3.50}$$

which - up to the last term - agrees with what was obtained from the supergravity discussion. In fact, the origin of such an extra term can be understood as stemming from the first subtlety we mentioned - the need to compactify before constructing E. Indeed, Eq. 3.32 is the reason we not only considered a four-form field strength F as present in the original eleven-dimensional action but also considered a seven-form field strength $\tilde{F}$ (dual to F). Had we focused only on the original theory, these two fields would not have been independent of each other (the Hodge star relates them). One can wonder if one can do the same thing with the metric and indeed the dualised metric - which is understood in the case of linearised gravity [37] - is the reason for the extra term. Taking a perturbation around a flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have $\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$, where ξ_ν is the Killing vector field. However, the symmetry transformation of the dualised version of this vector is automatically 0. Consequently, this is the reason why it did not appear in our previous discussion. Accordingly, we expect this part of the bundle not to generate symmetry transformations on the rest, which we will see is what happens later. [18]

In summary, an untwisted generalised vector on $U_{(i)}$ is given by:

$$V_{(i)} = v + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)}, \tag{3.51}$$

where $v \in TM$ (again the vector is the only globally defined tensor), $\omega_{(i)} \in \Lambda^2 T^* U_{(i)}$ $\sigma_{(i)} \in \Lambda^5 T^* U_{(i)}$, $\tau_{(i)} \in T^* U_{(i)} \otimes \Lambda^7 T^* U_{(i)}$. We can then see that the individual patchings found in Eq. 3.46, 3.47 and 3.49 all follow from the $E_{7(7)}$ action between patches:

$$V_{(i)} = e^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} \cdot V_{(j)}. \tag{3.52}$$

This gives a $GL(7,\mathbb{R}) \ltimes (\Omega^3_{cl} \ltimes \Omega^6_{cl})$ structure group and constrains the missing $\tau_{(i)}$ patching as:

$$\begin{aligned} \tau_{(i)} = \tau_{(j)} &+ j\mathrm{d}\Lambda_{(ij)} \wedge \sigma_{(j)} - j\mathrm{d}\tilde{\Lambda}_{(ij)} \wedge \omega_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge i_v \mathrm{d}\tilde{\Lambda}_{(ij)} \\ &+ \frac{1}{2} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \frac{1}{6} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge i_v \mathrm{d}\Lambda_{(ij)}, \end{aligned} \tag{3.53}$$

where the "j" notation defined in the appendix was used.

As the structure group is a semidirect product group, it (and the generalised tangent bundle by extension) can be written as a sequence as seen before. However, to account for the choice made in Eq. 3.52 - 3.53, the sequence is done slightly less straightforwardly than before as:

$$\begin{aligned} 0 \to \Lambda^2 T^* M \to E'' \to TM \to 0, \\ 0 \to \Lambda^5 T^* M \to E' \to E'' \to 0, \\ 0 \to T^* M \otimes \Lambda^7 T^* M \to E \to E' \to 0, \end{aligned} \tag{3.54}$$

and again we see that an element of the generalised tangent bundle is isomorphic to the expression given in Eq. 3.51 because we considered an exact sequence of vector spaces. Once more, $\Lambda_{(ij)}$ must satisfy some consistency requirements on $U_{(i)} \cap U_{(j)} \cap U_{(k)}$ and $U_{(i)} \cap U_{(j)} \cap U_{(k)} \cap U_{(l)}$ given respectively by[19]:

$$\begin{aligned} \Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)}, \\ \Lambda_{(jkl)} - \Lambda_{(ikl)} + \Lambda_{(ijl)} - \Lambda_{(ijk)} = \mathrm{d}\Lambda_{(ijkl)}, \end{aligned} \tag{3.55}$$

which means again that mathematically, $\Lambda_{(ij)}$ define a connection on a gerbe. Similarly, $\tilde{\Lambda}_{(ij)}$ also satisfy some consistency requirements which we do not give as their specific form is not particularly relevant for the other chapters (and because they go up to eight patches' intersections).

Finally, in order to allow for the warp factor when doing the dimensional reduction, $E_{7(7)}$ must

[18]Note that this discussion is irrelevant for $E_{d(d)}$ with $d \leq 6$ as no seven-form exists in this context. Conversely, the fact that the"usual" construction starts to break down for $d = 7$ hints as to why generalised geometry does not work for $d \geq 8$.

[19]Along again some further requirements if the flux is quantised, which we will not give.

be extended to $E_{7(7)} \times \mathbb{R}^+$, the last factor being known as the "trombone symmetry" (which in addition enables to find an isomorphism between a generalised vector and a sum of conventional vectors and forms as we already did) [38]. Then, a generalised vector obeys the same relations as Eq. 3.21, except $M \in E_{7(7)} \times \mathbb{R}^+$, hence justifying the appellation of a generalised vector, and the generalised structure bundle can be defined as a sub-bundle of the frame bundle for E:

$$\tilde{F} = \{(x, \{\hat{E}_A\}) \mid x \in M_7\}, \tag{3.56}$$

where $\{\hat{E}_A\}$ is a basis defined similarly to the previous case.

The adjoint bundle of $E_{7(7)} \times R^+$ can be decomposed under $GL(7, \mathbb{R})$ to give:

$$\mathrm{ad}\tilde{F} = \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 T^*M \oplus \Lambda^6 TM, \tag{3.57}$$

corresponding to the generalised adjoint tensor:

$$R = c + r + a + \tilde{a} + \alpha + \tilde{\alpha} \tag{3.58}$$

which is indeed 134-dimensional[20]. Listing each possible pairing between elements of R and elements of V using the same $\mathfrak{gl}(d, \mathbb{R})$ action as before and seeing whether the result lies in TM, $\Lambda^2 T^*M$, $\Lambda^5 T^*M$ or $(T^*M \otimes \Lambda^7 T^*M)$, the components of $V' = R \cdot V$ are found to be:

$$\begin{aligned} v' &= cv + r \cdot + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\ \omega' &= c\omega + r.\omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\ \sigma' &= c\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\ \tau' &= c\tau + r \cdot \tau - j\tilde{a} \wedge \omega + ja \wedge \sigma. \end{aligned} \tag{3.59}$$

Acting by $a + \tilde{a}$, we obtain:

$$\begin{aligned} e^{a+\tilde{a}} \cdot V &= (1 + (a + \tilde{a}) + \frac{1}{2}(a + \tilde{a})^2 + ...)V \\ &= V + (v \lrcorner a + (v \lrcorner \tilde{a} + a \wedge \omega) + (-j\tilde{a} \wedge \omega + ja \wedge \sigma)) \\ &+ \frac{1}{2}(a \wedge v \lrcorner a + (-j\tilde{a} \wedge (v \lrcorner a) + ja \wedge (v \lrcorner \tilde{a} + a \wedge \omega)) + 0. \end{aligned} \tag{3.60}$$

So we see that again the action is nilpotent (except now square terms still matter).

The action of $e^{\alpha+\tilde{\alpha}}$ can be found by the same reasoning (cubic terms also cancel), so that a general element of $E_{7(7)} \times R^+$ is given by:

$$M \cdot V = e^{\lambda} e^{\alpha+\tilde{\alpha}} e^{a+\tilde{a}} m \cdot V, \tag{3.61}$$

where m gives the standard $GL(7, \mathbb{R})$ action on tensors while e^{λ} gives the R^+ scaling.[21] We will

[20] The dimension of $E_{7(7)}$ is 133.

[21] Sometimes, one is not interested in this scaling. If that is the case, c is set to $\frac{1}{9-d} r^a{}_a$, which removes one

later need to evaluate the commutator between two sections of the adjoint bundle. The reasoning is roughly similar, the components of $[R, R']$ being given in Eq. C7 of [39].

There is another way to construct generalised vectors, which will be used later. The $E_{7(7)}$ group is a subgroup of $Sp(56, \mathbb{R})$, meaning that the symplectic product Ω on the fundamental **56** representation is left invariant:

$$\Omega(V, W) = \Omega_{AB} V^A W^B = v^{ab} w'_{ab} - v'_{ab} w^{ab}, \tag{3.62}$$

where **56** was decomposed by the $SL(8, \mathbb{R})$ subgroup of $E_{7(7)}$ such that:

$$\begin{aligned} \mathbf{56} &= \mathbf{28} + \mathbf{28}', \\ W &= \Lambda^2 V \oplus \Lambda^2 V^*, \\ V^A &= \begin{pmatrix} v^{aa'} \\ v'_{aa'} \end{pmatrix}. \end{aligned} \tag{3.63}$$

Here W refers to a 56-dimensional vector space (i.e. $1 \leq A \leq 56$) and V to the module of the $SL(8, \mathbb{R})$ representation (i.e. $1 \leq a, a', b, b' \leq 8$).

So far, we just insisted that $E_{7(7)}$ is some subgroup of the symplectic group. To fully specify it, it is sufficient to know that its action on W leaves invariant a particular quartic map q defined as:

$$\begin{aligned} q(V, V, V, V) = q_{ABCD} V^A V^B V^C V^D = v^{ab} v'_{bc} v^{cd} v'_{da} \\ -\frac{1}{4} v^{ab} v'_{ab} v^{cd} v'_{cd} + \frac{1}{96} (\epsilon_{abcdefgh} v^{ab} v^{cd} v^{ef} v^{gh} + \epsilon^{abcdefgh} v'_{ab} v'_{cd} v'_{ef} v'_{gh}). \end{aligned} \tag{3.64}$$

This means that alternatively to specifying generalised vectors by the corresponding $E_{7(7)} \times R^+$ version of Eq. 3.21, one can require the invariance of q and Ω to define them. Note that as usual, the specifics of these invariants depend on whether Type II or eleven-dimensional supergravity is compactified and the dimension of the compact space. [36]

Finally note that an operation that maps a generalised vector and a generalised covector to a generalised adjoint tensor can be defined as:

$$\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\tilde{F}, \tag{3.65}$$

which we will need later on (and whose explicit expression is given in Appendix C of [28]). Physically, in the same way that the adjoint bundle is where B and its conjugate were living in the previous case, we see that $A_{(i)}$, $\tilde{A}_{(i)}$ and their conjugate can now also be thought of as sections of the adjoint bundle (with again the $(TM \otimes T^*M)$ term coming from $gl(\mathrm{d}, \mathbb{R})$).

degree of freedom in the correct manner.

3.2 Dorfman Derivative and Courant bracket

3.2.1 $O(d,d)\times R^+$ Generalised Geometry

Taking $V=v+\lambda, V'=v'+\lambda', V''=v''+\lambda''\in\Gamma(E)$, the NSNS symmetry is equivalently captured by the following algebra:

$$\begin{aligned} v'' &= [v,v'],\\ \mathrm{d}\lambda'' &= \mathcal{L}_v\mathrm{d}\lambda' - \mathcal{L}_{v'}\mathrm{d}\lambda \end{aligned} \tag{3.66}$$

which, can be integrated following the choice of Eq. 3.15. Explicitely:

$$\begin{aligned} \mathrm{d}\lambda'' = \mathrm{d}(i_v\mathrm{d}\lambda') + i_v(\mathrm{d}\mathrm{d}\lambda') - (\mathrm{d}(i_{v'}\mathrm{d}\lambda) + i_{v'}(\mathrm{d}\mathrm{d}\lambda))\\ = \mathrm{d}(i_v\mathrm{d}\lambda' - i_{v'}\mathrm{d}\lambda)\\ \Rightarrow \lambda'' - (i_v\mathrm{d}\lambda' - i_{v'}\mathrm{d}\lambda) = \mathrm{d}f. \end{aligned} \tag{3.67}$$

where Cartan's magic formula was used in the first line and the nilpotency and linearity of the exterior derivative in the second. Taking $\mathrm{d}f=0$ amounts to the patching choice made earlier and leads to the following definition of the Dorfman (or "generalised Lie") derivative:

$$v''+\lambda'' = [v,v'] + \mathcal{L}_v\lambda' - i_{v'}\mathrm{d}\lambda := L_V V'. \tag{3.68}$$

This result/definition justifies the terminology of "generalised Lie derivative": in conventional geometry, the Lie derivative measures how much a tensor changes due to diffeomorphism, which is exactly how it is defined here (replacing diffeomorphism by its generalised definition that encompasses gauge transformations as we saw).

The key property of the Dorfman derivative is that it preserves the canonical $O(d,d)$ metric introduced in Eq. 3.18 in the sense that:

$$\eta(L_V U, W) + \eta(U, L_V W) = \mathcal{L}_v\eta(U,W). \tag{3.69}$$

for $V,U,W\in\Gamma(E)$.

This means that the generalised Lie derivative can be extended, similarly[22] to the conventional Lie derivative, to a generalised tensor of weight p, W:

$$\begin{aligned} L_V W^{M_1\dots M_n} = V^N\partial_N W^{M_1\dots M_n} + (\partial^{M_1}V^N - \partial^N V^{M_1})W_N{}^{M_2\dots M_n} + \dots\\ +(\partial^{M_n}V^N - \partial^N V^{M_n})W^{M_1\dots M_{n-1}}{}_N + p(\partial_N V^N)W^M, \end{aligned} \tag{3.70}$$

where indices were contracted using η_{MN} and we used:

$$\partial_M = \begin{cases} \partial_\mu & \text{if } M=\mu \\ 0 & \text{if } M=\mathrm{d}+\mu. \end{cases} \tag{3.71}$$

[22]This is the same component form as in conventional geometry with the exception that $\mathfrak{gl}(\mathrm{d})$ is replaced by the adjoint in $\mathfrak{o}(\mathrm{d,d})\oplus\mathbb{R}$, so $L\in E\otimes \mathrm{ad}$ This should dispel any unease to call L the generalised Lie derivative.

Note also that the Dorfman derivative satisfies Leibniz's rule in the form:

$$L_V(L_{V'}U) - L_{V'}(L_V U) = L_{L_{V'}V}U, \tag{3.72}$$

for $V, V', U \in \Gamma(E)$.

The generalised Lie derivative possesses nonetheless one major difference compared to its conventional geometry counterpart: it is not antisymmetric (when acting on a generalised vector). Formally, this means that $L_V V'$ does not define a Lie algebroid, but a Courant algebroid. The failure of antisymmetry is captured by the Courant bracket, which takes the form:

$$\begin{aligned}
[[V, V']] &:= \frac{1}{2}(L_V V' - L_{V'}V) \\
= [v, v'] + \mathcal{L}_v\lambda' - \mathcal{L}_{v'}\lambda - \frac{1}{2}\mathrm{d}(i_v\lambda' - i_{v'}\lambda) &= L_V V' - \frac{1}{2}dV, V' \\
= (V^N\partial_N V'^M - V'^N\partial_N V^M - \frac{1}{2}(V_N\partial^N V'^M - V'_N\partial^N V^M))\partial_M &
\end{aligned} \tag{3.73}$$

Note however that for $V = \hat{E}_A$ and $V' = \hat{E}_B$, the derivative terms of the second line vanish, resulting in:

$$L_{\hat{E}_A}\hat{E}_B = [[\hat{E}_A, \hat{E}_B]]. \tag{3.74}$$

Finally, the Courant bracket is invariant under G_{NS}, the structure group of the NSNS sector of Type II supergravity with our choice of integration constant.

3.2.2 $E_{d(d)} \times R^+$ Generalised Geometry

Again the bosonic symmetries are captured by the following algebra:

$$\begin{aligned}
v'' &= [v, v'], \\
\mathrm{d}\omega'' &= \mathcal{L}_v\mathrm{d}\omega' - \mathcal{L}_{v'}\mathrm{d}\omega, \\
\mathrm{d}\sigma'' &= \mathcal{L}_v\mathrm{d}\sigma' - \mathcal{L}_{v'}\mathrm{d}\sigma - \mathrm{d}\sigma' \wedge \mathrm{d}\sigma, \\
\mathrm{d}\tau'' &= \mathcal{L}_v\mathrm{d}\tau' - j\mathrm{d}\sigma \wedge \mathrm{d}\omega - j\mathrm{d}\omega' \wedge \mathrm{d}\sigma.
\end{aligned} \tag{3.75}$$

Integrating with the appropriate choice, we can define the $E_{d(d)} \times R^+$ version of the generalised Lie derivative in the same way as before:

$$\begin{aligned}
L_V V' &:= v'' + \omega'' + \sigma'' + \tau'' \\
= \mathcal{L}_v v' + (\mathcal{L}_v\omega' - i_{v'}\mathrm{d}\omega) + (\mathcal{L}_v\sigma' - i_{v'}\mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega) &+ (\mathcal{L}_v\tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma), \\
L_V f &:= \mathcal{L}_v f,
\end{aligned} \tag{3.76}$$

for a function f. The last equation is the usual requirement that all derivatives acting on a function must agree. Similarly to Eq. 3.70, the adjoint action (defined in Eq. 3.65) is used, which we can take to obtain the Lie derivative in the succinct manner:

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N. \tag{3.77}$$

The failure of antisymmetry is captured by the exceptional Courant bracket:

$$\begin{aligned} [[V,V']] &:= \frac{1}{2}(L_V V' - L_{V'} V) \\ &= [v,v'] + \mathcal{L}_v\omega' - \mathcal{L}_{v'}\omega - \frac{1}{2}\mathrm{d}(i_v\omega' - i_{v'}\omega) + \mathcal{L}_v\sigma' - \mathcal{L}_{v'}\sigma - \frac{1}{2}\mathrm{d}(i_v\sigma' - i_{v'}\sigma) \\ &+\frac{1}{2}(\omega\wedge\mathrm{d}\omega' - \omega'\wedge\mathrm{d}\omega) + \frac{1}{2}(\mathcal{L}_v\tau' - \mathcal{L}_{v'}\tau) + \frac{1}{2}(j\omega\wedge\mathrm{d}\sigma' - j\omega'\wedge\mathrm{d}\sigma) - \frac{1}{2}(j\sigma'\wedge\mathrm{d}\omega - j\sigma\wedge\mathrm{d}\omega'). \end{aligned} \tag{3.78}$$

This bracket is also invariant under the structure group of the bosonic symmetries (of 11 dimensional supergravity) $GL(d,\mathbb{R}) \ltimes (\Omega^3_{cl} \ltimes \Omega^6_{cl})$ and when evaluated on the frame basis, Eq. 3.74 remains valid for the same reason.

Finally, it should be emphasised that the fact that a generalisation of the Lie derivative in the form of Eq. 3.77 can be defined is non-trivial as Eq. 3.77 is not in general a generalisation of a tensor. For instance, it would not be possible to do the same for $Sp(2n,\mathbb{R})$ since the adjoint action is symmetric and is hence not a tensor. This is formalised in the notion of g-algebroids and their full classification is still an open question. This is why the fact that it can be done for $O(d,d)\times\mathbb{R}^+$ and $E_{d(d)}\times\mathbb{R}^+$ is deemed to be an important and non-trivial step.

3.3 Generalised Metric

3.3.1 $O(d,d)\times R^+$ Generalised Geometry

It was shown in Section 2.1 why a Riemannian metric at a point belongs to the quotient $\frac{GL(d,\mathbb{R})}{O(d)}$, where $O(d)$ is the[23] maximal compact subgroup of $GL(d,\mathbb{R})$. The metric G then transforms a new metric G' as:

$$G' = gG,\ g\in Gl(d,\mathbb{R}) \text{ such that } G = hG,\ h\in O(d). \tag{3.79}$$

Similarly, a Riemannian $O(d,d)\times\mathbb{R}^+$ generalised metric at a point can be defined as an element of $\frac{O(d,d)\times\mathbb{R}^+}{O(d)\times O(d)}$, where $O(d)\times O(d)$ is a maximal compact subgroup of $O(d,d)\times\mathbb{R}^+$. For a pseudo-Riemannian equivalent of $(p,d-p)$ signature, $O(d)\times O(d)$ can be replaced by $O(p,d-p)\times O(d-p,p)$. Counting the number of dimensions suggests that all the fields of the NSNS sector of Type II supergravity can potentially be unified in such a generalised metric:

$$\begin{aligned} \dim\left(\frac{O(d,d)\times\mathbb{R}^+}{O(d)\times O(d)}\right) &= \frac{2d(2d-1)}{2} + 1 - 2\left(\frac{d(d-1)}{2}\right) = d^2+1 \\ &= \frac{d(d+1)}{2} + \frac{d(d-1)}{2} + 1 = \dim(g) + \dim(B) + \dim(\phi). \end{aligned} \tag{3.80}$$

Formally, it can be shown to be equivalent to choosing a positive definite subbundle such that

[23] For any Lie group, the Cartan-Iwasawa-Malcev theorem states that maximal compact subgroups are essentially unique - meaning here unique up to conjugation. [40]

its orthogonal complement is negative definite and $E = C^+ \oplus C^-$ [3]. Given this choice, the generalised metric G is required to be compatible with the canonical metric in the sense that [41]:

$$G(V, V') = \langle V, V' \rangle |_{C_+} - \langle V, V' \rangle |_{C_-}, \; G^2 = \mathbf{1}. \tag{3.81}$$

In practice [42], up to a factor given by the canonical metric, taking $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the compatibility can be rephrased as:

$$\eta^{AB} G_{BC} \eta^{CD} = G^{AD}. \tag{3.82}$$

This implies that $a = a^T$, $d = d^T$, $b = c^T$. Taking $g = \frac{1}{2} d^{-1}$ and $B = d^{-1}c$ (which are indeed respectively symmetric and antisymmetric), we have:

$$G_{MN} = \frac{1}{2} \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}_{MN}, \; \Phi = e^{-2\phi}\sqrt{-g}. \tag{3.83}$$

Hence, the generalised metric indeed unifies the NSNS sector fields. Note that this can be rewritten as:

$$G = (e^B)^T G_0 e^B \tag{3.84}$$

where $G_0 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ and $e^B = \begin{pmatrix} \mathbf{1} & 0 \\ B & \mathbf{1} \end{pmatrix}$.

Equivalently, a generalised metric can be defined directly on a Courant algebroid as a subbundle whose rank is half the rank of E such that the inner product restricted on that subbundle is positive-definite. Since the inner product is already non-degenerate and symmetric, we see that generalised metrics closely resemble the usual Riemannian metrics (without even having to see them as G-structures) [43].

3.3.2 $E_{d(d)} \times R^+$ Generalised Geometry

The double cover[24] of the maximal compact subgroup of $E_{7(7)} \times R^+$ is $\tilde{H} = SU(8)$. The $E_{7(7)} \times R^+$ generalised metric can therefore be defined as an element of $\frac{E_{7(7)} \times R^+}{SU(8)}$ at each point and again, the following counting argument suggests that it can unify the bosonic fields:

$$\begin{aligned} &\dim\left(\frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)}\right) = 133 + 1 - (8^2 - 1) \\ &= \frac{7(7+1)}{2} + 2\frac{7.6.5}{3.2} + 1 = \dim(g) + \dim(A) + \dim(\tilde{A}) + \dim(\Delta). \end{aligned} \tag{3.85}$$

Similarly[25] to Eq. 3.83- 3.84, we can start with the diagonal metric

$$G_0(V, V) = v^2 + \frac{1}{2}\omega_{ab}\omega^{ab} + \frac{1}{5!}\sigma_{a_1 \dots a_5}\sigma^{a_1 \dots a_5} + \frac{1}{7!}\sigma_{a, a_1 \dots a_7}\sigma^{a, a_1 \dots a_7}. \tag{3.86}$$

[24] Assuming M is a spin manifold, this is to account for fermions.

[25] We are being very schematic here because the precise details go beyond the scope of this thesis and the main point is that the generalised metric unifies the bosonic fields.

The full metric can then be found by acting on the left and the right by a subroup of $E_{7(7)} \times R^+$ as in Eq. 3.84 [27], [44].

Hence from the point of view of the non-compact manifold, the scalars are given by the metric G_{MN}, which matches the comment made in Section 2.3 where it was said that the scalars of the theory arranged themselves in a coset $\frac{G}{H}$, where G is the global symmetry group of the theory and H is the maximal compact subgroup of G.[26] In general we will also have one-forms and two-forms which are given in summary by [16]:

$$\begin{aligned} G_{MN}(x,y) &\in \Gamma(S^2E^*), \\ \mathcal{A}_\mu{}^M(x,y) &\in \Gamma(T^*X \otimes E), \\ \mathcal{B}_{\mu\nu}{}^{MN}(x,y) &\in \Gamma(\Lambda^2 T^*X \otimes N), \end{aligned} \tag{3.87}$$

where X denotes the non-compact manifold and N is a specific subgroup of S^2E given by the $\mathbf{133_2}$ representation for $d = 7$ [28].

3.4 Generalised Connection and Torsion

A generalised connection D is a first-order linear differential operator, given on $V = V^a \hat{E}_A \in \Gamma(E)$ by:

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B. \tag{3.88}$$

The M index indicates that Ω is a generalised covector while the A and B indices mean the domain is $\mathrm{ad}\tilde{F}$.

As in conventional geometry, this is too general to be used in later calculations. One can however constrain the connection by imposing some compatibility condition. A (generalised) connection is said to be compatible with a (generalised) G-structure P (where P is a subbundle of $\tilde{F}$ with fibre G) if

$$DK_{(a)} = 0,\ K_{(a)}|_x \in \frac{G'}{G}, \tag{3.89}$$

where G' is the group of the (generalised) frame bundle considered (so here $G' = O(d,d) \times R^+$ or $G' = E_{7(7)} \times R^+$). Then, a connection compatible with a G-structure cannot rotate a vector outside of that G-structure, which renders the terminology of "compatibility" transparent.[27]

The first obvious requirement is the compatibility with G', meaning that $D\eta = 0$. For instance, for the $O(d,d)$ canonical metric, this imposes the condition that $\Omega_M{}^{AB} = -\Omega_M{}^{BA}$. Similarly to general relativity, an obvious second compatibility requirement is that of the maximal compact[28] subgroup of G'. This is the metric connection which is obtained by requiring $DG = 0$ and $D\Phi = 0$.

[26] A gauging should be done then by taking a subgroup of $E_{d(d)} \times R^+$.
[27] Connections can also be directly defined on a principal bundle, which will agree with the connections as defined here given these are compatible with the G-structure, which also justifies the name.
[28] Or its double cover.

The generalised torsion T of a generalised G-compatible connection D can be defined as[29]:

$$T(V) = L_V^D - L_V, \tag{3.90}$$

where L_V^D indicates that the ∂ have been replaced by some connection (not necessarily metric compatible unless specified). Since for a compatible connection, the first term is zero when evaluated on an element of a generalised frame bundle, the torsion of a compatible connection is also given by:

$$T_A{}^B{}_C \hat{E}_B = -L_{\hat{E}_A} \hat{E}_C, \tag{3.91}$$

where $\hat{E}_A, \hat{E}_B, \hat{E}_C$ are vectors of a generalised frame.

Since at a point, $L, L^D \in E \otimes \mathrm{ad}\tilde{F}$, the torsion at a point will in general live in a subspace W, which means that it will be in an irreducible representation of $E \otimes \mathrm{ad}\tilde{F}$. For the $O(d,d) \times R^+$ case, $W = \Lambda^3 E \oplus E$ while $E_{7(7)} \times R^+$, the representation is $\mathbf{912_{-1}} + \mathbf{56_{-1}}$. Physically, this matches the embedding tensor (with the trombone symmetry). This correspondence continues to be exact for $4 \leq d \leq 6$. A connection whose torsion vanishes is called torsion-free.[30]

A generalised $O(d,d) \times R^+$ ($E_{d(d)} \times R^+$) Levi-Civita connection hence can finally be defined: it is an $O(d,d) \times R^+$ ($E_{d(d)} \times R^+$) and metric compatible connection that is generalised torsion-free. Contrarily to a conventional Levi-Civita connection, a generalised torsion-free metric compatible connection still always exists but is not unique (except for $E_{3(3)}$), which was proved in [28].

As we just saw, the torsion is defined from a choice of two things: a structure $\tilde{P}_G$ and a connection D. However one can ask if some subspace of the torsion space is independent on the latter choice. This subspace in fact exists. This is the idea of the intrinsic torsion, which can be found as follows [45].

For some (generalised) compatible connections D, D' and their (generalised) torsions $T^D, T^{D'}$, we have:

$$D' - D \in \Gamma(K_G), \quad T^D \in \Gamma(W), \tag{3.92}$$

where $K_G = E^* \otimes \mathrm{ad}\tilde{P}_G$[31] - representing the ambiguous part of the connection - and W[32] is the space of torsions. A map $\tau : K_G \to W$ can be defined such that:

$$\tau(D' - D) = T^{D'} - T^D. \tag{3.93}$$

Then, $U_G := \ker \tau \subset K_G$ is the space of compatible connections that lead to a set given torsion and $W_G := \mathrm{im}\, \tau \subset W$ and $W_{int} := \frac{W}{W_G}$ is the space of intrinsic torsions that we were looking for. This also means that W_{int} gives a way of classifying the structure P_G. Alternatively, a non-zero W_{int} also indicates that there is no torsion-free connection and if $W_{int} = 0$, $\tilde{P}_G$ is called torsion-free (or integrable) G-structure. For instance, an H_d-structure is torsion-free. Finally, if $G \subset H_d$, there

[29]This is analogous to a property of torsion in conventional geometry which is taken here as the starting point.

[30]Note that any conventional connection can be uplifted to a generalised connection, but a torsion-free connection will in general uplift to a torsionful connection in the generalised sense of Eq. 3.88, as illustrated in [26].

[31]$K_G = T^*M \otimes \mathrm{ad}\tilde{P}_G$ for Riemannian geometry.

[32]$W = TM \otimes \Lambda^2 T^*M$ for the usual case and was already given for the two generalised geometries studied here.

exists a unique decomposition provided by the generalised metric:

$$W = W_G \oplus W_{int}, \quad K_G = W_G \oplus U_G. \tag{3.94}$$

As an example, we can take a certain embedding of $G = Sp(6) \times SU(2) \subset O(6,6)$. Since $\frac{12!}{3!9!} = 220$, we can decompose $W = \mathbf{220} + \mathbf{12}$ under the particular embedding of G given by LieArt to obtain: $2(\mathbf{6},\mathbf{4}) + (\mathbf{6},\mathbf{4}) + (\mathbf{64},\mathbf{2}) + (\mathbf{14'},\mathbf{4})$. Since the adjoint representation of $Sp(6)$ is $\mathbf{21}$, we can decompose $K_G = (\mathbf{21},\mathbf{1}) + (\mathbf{1},\mathbf{3})$ under G to obtain $2(\mathbf{6},\mathbf{4}) + (\mathbf{6},\mathbf{4}) + (\mathbf{64},\mathbf{2}) + (\mathbf{56},\mathbf{2})$. Assuming no kernel[33], one can read off using Eq. 3.94 that this structure has intrinsic torsion, specifically that:

$$\begin{aligned} W_{Sp(6)\times SU(2)} &= 2(\mathbf{6},\mathbf{4}) + (\mathbf{6},\mathbf{4}) + (\mathbf{64},\mathbf{2}), \\ U_{Sp(6)\times SU(2)} &= (\mathbf{56},\mathbf{2}), \\ W_{int} &= (\mathbf{14'},\mathbf{4}). \end{aligned} \tag{3.95}$$

In conventional geometry, the components of the intrinsic can be calculated by acting on the invariant tensors with the Levi-Civita connection. Indeed, since it is unique, a generic metric and $G \subset O(d)$-structure compatible connection ∇ can be written as $\nabla_{LC} - K$, where the torsion is given by antisymmetrising the covector indices of K. So, as all of the torsion lies in K, it can be identified with the intrinsic torsion and we have - taking the G-structure to be defined by the invariant tensors Ξ_i - :

$$\nabla \Xi_i = 0 \Rightarrow \nabla^{LC}\Xi_i = K\Xi_i, \tag{3.96}$$

$(T_{int})_{mn}{}^p = K_n{}^p{}_m - K_m{}^p{}_n$. Hence trivially, a $(G \subset O_d)$-structure is torsion-free if and only if $\nabla^{LC}\Xi_i = 0$.

In generalised geometry, a singlet intrinsic torsion can also be found as follows if one of the invariant generalised tensors Ξ_i can be given by some generalised vector K_A:

$$T^D_{int}(K_A) \cdot \Xi_i = -L_{K_A}\Xi_i, \tag{3.97}$$

which follows from Eq. 3.90 as well as the fact that it acts on singlets and $L^D_{K_A}Q_i = 0$ for a singlet intrinsic torsion since $D\Xi_i = 0$. Eq. 3.97 tells us that the singlet T_{int} is a singlet of $\mathrm{ad}\tilde{F}$ whose domain is a generalised vector, which will become important when gauging the truncated theory.

Requiring a structure to be torsion-free leads to natural conditions. For instance, an almost symplectic structure is equivalent to a globally defined two-form. It is torsion-free if this form is also closed. The structure is then called symplectic. An identity structure is equivalent to the existence of a globally defined frame. From Eq. 3.91, it is torsion-free if these commute which implies the existence of coordinates locally. Physically, torsion-free structures give fluxless supersymmetric backgrounds.

The next step would be to define a generalised notion of curvature. It can be proved however that its naive definition would not lead to a generalised tensor [28]. This is not so important as, for

[33]We will assume this in the following for simplicity. An explicit calculation showing how to check this can be found in Appendix F of [39].

a H_d-compatible torsion-free connection, the generalised Ricci tensor R_{AB} and generalised Ricci scalar R - which we do not give here as they will not be so relevant later - are well-defined. In addition, in the language of generalised geometry, the bosonic part of supergravity is none other than generalised gravity:

$$S_B = \int_{\text{vol}_G} R. \tag{3.98}$$

Finally, we have avoided any mention of fermions but in fact these also fit naturally inside representations of $\tilde{H}_d$, the double cover of the maximal compact subgroup of G: one is called the spinor bundle $\mathcal{S}$ and gives the number of supersymmetry parameters and another is the gravitino bundle which gives gravitino fields. Therefore, generalised geometry also provides with simple reformulations of the fermionic equations of motion and of the supersymmetry variations.

3.5 Type II Supergravity

The full Type II A/B supergravities are also given in terms of exceptional generalised geometry. We do not explained the results of this section but merely point at the most basic difference - that is, at the level of the generalised tangent bundle - between the exceptional general geometries of Type IIA, IIB and eleven-dimensional supergravities. When needed, the other necessary formulas will then be indicated. Type IIA is given by a dimensional reduced M-theory. This dimensional reduction is done by taking the subgroup $Gl(d-1,\mathbb{R}) \subset Gl(d,\mathbb{R})$ when defining $E_{d(d)}$[34]. This implies that the generalised tangent bundle takes the form [28]:

$$E_{\text{Type IIA}} \cong TM \oplus T^*M \oplus \Lambda^{even}T^*M \oplus \Lambda^5T^*M \oplus (T^*M \otimes \Lambda^6T^*M), \tag{3.99}$$

where now this concerns a compactification on a $(d-1)$-dimensional manifold, contrarily to before. To recover Type IIB, we need to take $GL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, before recombining into $GL(d-1,\mathbb{R}) \times SL(2,\mathbb{R})$ such that:

$$E_{\text{Type IIB}} \cong TM \oplus T^*M \oplus \Lambda^{odd}T^*M \oplus \Lambda^5T^*M \oplus (T^*M \otimes \Lambda^6T^*M). \tag{3.100}$$

All the steps described previously in this section can subsequently be applied to these two cases.

[34] This comes from what was explained in the second way to construct generalised vectors in Section 3.1.2

4 Consistent Truncations

4.1 Motivation

Some of the motivations to find consistent truncations of string theory and M-theory relating to the swampland were alluded to in the introduction. The advent of the ADS/CFT correspondence has given even more pretext for its study.

Maldacena's orginal formulation of the conjecture considers two theories: on the one hand Type IIB string theory with coupling g_s and string length $\sqrt{\alpha'}$ compactified to AdS_5 on a five-sphere whose radius is L with N units of Ramond-Ramond fluxes, on the other hand $\mathcal{N} = 4$ Super Yang-Mills theory - with coupling g_{YM} and gauge group $SU(N)$. These theories describe the same physics if we have:

$$g_{YM}^2 = 2\pi g_s = \frac{L^4}{2N\alpha'^2}. \tag{4.1}$$

This can be seen by considering the dynamics from both an open string and a closed string perspectives [46]. This case can be generalised, but remains a conjecture. Nevertheless, it is very useful as it relates a strongly-coupled theory to a weakly-coupled one (which can be calculated). On the AdS side, considering the full string theory is hard so it is desirable to keep only a finite subset of states. It might be tempting to construct examples in a bottom-up approach - that is, starting with solutions of a gravitational theory with some extra degrees of freedom and hope that it uplift to string or M theory. In general, this hope is not realised. Consistently truncating therefore seems like the only other option - that is, going in a top-down approach [47].

In the original Kaluza-Klein theory, the reduction is made on a circle whose isometry group is $U(1)$. This implies that fields can be expanded as a simple Fourier series and the mass of the non-zero modes is inversely proportional to the radius of the circle. Truncating to only the zero-mode came therefore from an effective field theory perspective: for a small radius, a separation of the mass scales is induced. It is then realised that the truncation is more than just that: it is in fact truly consistent with the higher-dimensional theory with no modification in the parameters.

This is hard to generalise specifically for an AdS background because of the AdS Distance Conjecture. It states that for any quantum gravity on AdS with cosmological constant Λ, "there exists an infinite tower of states with mass scale m which, as $\Lambda \to 0$, behaves (in Planck units) as $m \sim |\Lambda|^\alpha$, where α is a positive order-one number"[35] [48]. This conjecture implies that there is no separation of scales between the radii of the compact internal manifold and of AdS. This poses a problem in our case as it prevents us from following the same effective field theory approach as above.

A method that would enable to find directly consistent truncations from string or M-theory can hence be used to understand better strongly coupled quantum field theories through the AdS/CFT correspondence.

[35]The Strong AdS Distance Conjecture take $\alpha = \frac{1}{2}$, which is consistent with all the known cases.

4.2 Situation before Generalised Geometry

In dimensional reductions, we consider:

$$M_D = M'_d \times K_n, \tag{4.2}$$

where M_D is a D dimensional spacetime (usually $D = 10, 11$), M'_d is $d < D$ spacetime (usually $d = 4$ if one is directly interested in the world we see, $d = 5$ if one is using the ADS/CFT correspondence), K_n is an n-dimensional compact space and $D = d + n$.

The KK background metric $g_{\Lambda\Sigma}$ on M_D will then be:

$$g_{\Lambda\Sigma}(x, y) = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) & 0 \\ 0 & g^{(0)}_{mn}(y) \end{pmatrix}, \tag{4.3}$$

where $g^{(0)}_{\mu\nu}(x)$ is the background metric on M'_d and $g^{(0)}_{mn}(y)$ is the background metric on K_n.[36]

A scalar field $\phi(x, y)$ on M_D is then expanded as [49]:

$$\phi(x, y) = \sum_{q, I_q} \phi^{I_q}_q(x) Y^{I_q}_q(y) \tag{4.4}$$

where q labels the eigenvalue of the Laplacian and $Y^{I_q}_q(y)$ are the eigenfunctions of the Laplacian on the compact space corresponding to the eigenvalue. In general, the eigenvectors a Laplacian on K_n are representations of the isometry group of K_n, which are labeled here by I_q [50],[51].

A KK ansatz for the dimensional reduction means keeping only the zero eigenvalue in the expansion. Then, a gauge field $A_\Lambda(x, y)$ will split as $A_\mu(x)$, a d-dimensional vector from M'_d and $A_m(y)$, scalars as seen from M'_d.

More generally for $p + 1 \le d$, $A_{\Lambda_1 \ldots \Lambda_{p+1}}$ splits into $A_{\mu_1 \ldots \mu_{p+1}}$, $A_{\mu_1 \ldots \mu_k m_{k+1} \ldots m_{p+1}}$, ..., $A_{\mu_1 \ldots \mu_{p-n+1} m_{p-n+2} \ldots m_{p+1}}$ and a M_D spinor η_A will split as:

$$\eta_A(x, y) = \eta^I_M(x)\epsilon^I_i(y), \tag{4.5}$$

where the $\epsilon^I_i(y)$ are the Killing spinors, which can be thought of as "square roots" of the Killing vectors V^{AB}_ν (satisfying $D_\mu V^{AB}_\nu = 0$), in the sense that on a sphere for instance they satisfy [49]:

$$\begin{aligned} V^{AB}_\nu &= \epsilon^I \gamma_\nu \epsilon^I (\gamma^{AB})_{IJ} \\ D_\nu \epsilon^I_i &\propto (\gamma_\nu \epsilon^I)_i. \end{aligned}$$

In general, the Killing spinor is defined as obeying: $\delta_{susy}\epsilon_A = 0$. Hence, by the supersymmetric algebra, Killing spinors are massless. In doing a KK type of ansatz - i.e. keeping all the massless (or $n = 0$) modes -, one keeps all the Killing spinors and the amount of supersymmetry is unchanged after the dimensional reduction.

This procedure assumes that the KK reduction ansatz is valid, a fact which is wrong as one

[36] We emphasise by "background" that this does not concern the full fluctuating spin 2 field of the theory.

then needs to check if the truncation operated is consistent, i.e. if the modes set to 0 can still be 0 when looking at the equations of motion on M_D. If this is not true, we say that the modes we kept "source" the other modes and the truncation is not consistent. For example we could have, for a ϕ^3 coupling, the following equation of motion:

$$(\Box - m_q^2)\phi_q^{I_q}(x) = \phi_0^{I_0}(x)\phi_0^{I_0}(x)(...) \tag{4.6}$$

where we see that truncated modes are sourced, meaning it is inconsistent. An important exception is when K_n is a torus T^n. In such a case, it is possible to arrange the same spin fields into multiplets of some global symmetry group G. A KK ansatz will then be consistent and the Killing spinors trivial as $V_m = 1$ [49]. One way to render an inconsistent truncation a consistent one is by making a field redefinition of the type (for instance):

$$\phi'_q = \phi_q + a\phi_0^2 + ...,$$
$$\phi'_0 = \phi_0 + \sum_p q b_{pq}\phi_p\phi_q.$$

An example of a nonlinear KK ansatz is given by:

$$g_{\mu\nu}(x,y) = g_{\mu\nu}(x)\left(\frac{\det g_{mn}(x,y)}{\det g_{mn}^{(0)}}\right)^{-\frac{1}{d-2}}, \tag{4.7}$$

which enables to recover the d-dimensional Einstein action from the D-dimensional one. However, it is easy to see that such field redefinitions can quickly become cumbersome.

If K_n is non-trivial (such as a sphere S^n), the abelian Killing spinors of the torus compactification become non-abelian, giving a gauge group $H \subset G$. Such compactifications result in gauged supergravities, deformations of the ungauged supergravity by a gauge coupling, which was presented earlier. This adds a negative cosmological constant term, which means that the natural background of gauged supergravity is AdS.

Before the advent of generalised geometry, the most general sufficient condition to the existence of a consistent truncation was the "Scherk-Schwartz construction". It states that if there exists a global basis for TM satisfying an unimodular considition, then a consistent truncation exists. Such a global basis - meaning nowhere vanishing - is called a parallelisation and is equivalent to whether M admits a trivial structure group. In practice, this can also be done by finding $\dim(TM)$ left-invariant vector fields $\hat{e}_a \in \Gamma(TM)$ which, by definition of a Lie algebra, satisfy one:

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c, \tag{4.8}$$

where $f_{ab}{}^c$ are constants and $f_{ab}{}^b = 0$ [15].

Manifolds that admit such a "parallelisation" must possess a lot of structure and truncations can be found on group manifolds $\mathcal{G}$ as well as cosets $\frac{\mathcal{G}}{\Gamma}$, where Γ is a descrete subroup chosen so that the coset is compact [52].

While this is useful, it is only a sufficient condition, not a necessary one. In particular, reductions on S^4 for instance lie outside of this construction.

4.3 General Formalism using Generalised Geometry

Generalised geometry enables to systematically construct consistent truncations of a much broader kind than Scherk-Schwartz. However, since in form it follows closely a construction that can be done on the usual $Gl(d,\mathbb{R})$ frame bundle, we will start with it, following closely [16].

Any equation of motion is constructed from the field content as well as the n-derivatives of some combination of them, taken without loss of generality as the Levi-Civita connection.[37] A first natural condition is to demand that only singlets form the truncated field content. This follows from the fact that every term in a Lagrangian has to be invariant or alternatively from the equivalent requirement that both sides of the corresponding equation of motion (for instance of the form of Eq. 4.6) transform similarly. In the latter view, allowing some non singlets to be 0 while retaining others would imply (in general) that one side of the equation of motion does not transform ($0 \to 0$, corresponding to the truncated modes) while the other does, violating symmetry. This is however not enough since nothing so far forbids the derivative terms of singlets to source non-singlets. Eq. 3.96 gives immediately the required condition: demand that only singlets appear when decomposing the intrinsic torsion under G. In this way, the derivative of a singlet never sources a singlet and because of the Leibniz rule (obeyed by the Levi-Civita connection), this is also true for the derivative of a product of singlets. This means also that any power of ∇^{LC} always return singlets. In summary, there are two sufficient conditions for the existence of a consistent truncation on a manifold M with a G_S-structure:

1) Only retain fields that transform under G_S as singlet,

2) Only consider G_S-structures with a singlet intrinsic torsion (or torsion-free).

The Scherk-Schwartz immediately appears as a special case. Indeed, requiring the existence of a global basis on M is equivalent to requiring $G_S = \mathbf{1}$, i.e. the strongest constraint to put from the point of view of the G-structure of the manifold.

One can go further than the existence claim and construct in part the truncation. First, by splitting the degrees of freedom as explained in the previous section, the number of vector degrees of freedom (the metric gauge fields) will be given by the number of invariant one-forms η^a in Ξ_i. Explicitely, one construct them from their dual vectors $\hat{\eta}_a$ as $\mathcal{A}^a\hat{\eta}_a$. The same procedure applies to higher tensors as well. Constructing the truncated metric scalar fields is slightly more involved as they do not belong to $GL(d,\mathbb{R})$ but to the quotient $\frac{GL(d,\mathbb{R})}{O(d)}$, as we saw when defining the metric. However, the singlets of a quotient $\frac{A}{B}$ under $G_S \subset B \subset A$ are given by:

$$\frac{C_A(G_S)}{C_B(G_S)}, \tag{4.9}$$

where $C_A(B)$ denotes the commutant of $B \subset A$ inside A, i.e. all the elements whose commutator with an element of B lies in B. This can be understood by considering the algebra (i.e. locally). A vector X is a singlet if it is invariant under the adjoint action, i.e. $gXg^{-1} = X$, which illustrates

[37] Because any compatible connection can be rewritten as $\nabla^{LC} - K$ as discussed earlier.

- but not proves - why one is interested in the commutant of A in B to find the singlets under G_S. Immediately, we that the Lie algebra of the commutant group $C_A(B)$ will be given by singlets of $\mathrm{ad}\tilde{F}$, where $\tilde{F}$ is the frame bundle of A.

Here, $A = GL(d, \mathbb{R})$ and $B = O(d)$, so the truncated metric scalar belong to $\frac{C_{Gl(d,\mathbb{R})}(G_S)}{C_{O(d)}(G_S)}$. Note that if $O(d) \subset G_S$, one can try to rescale the dimension d until $G_S \subset O(nd)$. For instance in the Scherk-Schwartz construction, the metric scalars belong then to $\frac{Gl(d,\mathbb{R})}{O(d)}$ since all group elements commute with the identity.

Finally, for invariant one-forms $\hat{\eta}_a$, the conventional geometry equivalent of Eq. 3.96 implies that:

$$[\hat{\eta}_a, \hat{\eta}_b] = f_{ab}{}^c \hat{\eta}_c, \tag{4.10}$$

where $f_{ab}{}^c$ are completely determined by the intrinsic torsion. This means that the gauging of the truncated theory is completely determined by the intrinsic torsion.

In the previous chapter, we showed that all the bosonic fields of eleven dimensional supergravity compactified on a seven-dimensional compact manifold can be seen as generalised tensors of $E_{7(7)} \times Gl(4, \mathbb{R})$. This can be extended to a $d \leq 7$ dimensional compact manifold and to Type II supergravity. This means that the previous statement can be enlarged to the case of generalised G_S-structures where $G_S \subset H_d$, where H_d is the maximal compact subgroup of $E_{d(d)}$. Explicitly, it was showed in [16] that there always exists a consistent truncation of Type II or eleven-dimensional supergravity if the compact manifold with a generalised G_S-structure has only constant singlet generalised intrinsic torsion and if the bosonic fields are expanded in terms of Q_i, the generalised invariant tensors defined by G_S. The truncation of the fermionic fields is done by lifting $G_S \subset H_d$ to $\tilde{G}_S \subset \tilde{H}_d$, where $\tilde{H}_d$ is the double cover of H_d, and by expanding again the fermionic field in therm of $\tilde{G}_S$ singlets.

This can work because the generalised intrinsic torsion plays the same role as the conventional intrinsic torsion as was shown earlier. There is however one subtlety, which is that the generalised Levi-Civita connection is not unique, although supergravity depend only certain unique projections. This means that again a generalised singlet intrinsic torsion will forbid any derivative appearing in the supergravity equations of motion to source non-singlets.

This procedure gives again the structure of the truncated theory, starting from the degrees of freedom summarised in Eq. 3.87. The reasoning is the same: the generalised invariant vectors and two-forms - called respectively $K_{\mathcal{A}}$ and J_Σ - span respectively $\mathcal{V}$ of $\Gamma(E)$ and $\mathcal{B}$ of $\Gamma(N)$, so that we have:

$$\begin{aligned} h^I(x) \in \mathcal{M}_{\text{scal}} &= \frac{C_{E_{d(d)}}(G_S)}{C_{H_d}(G_S)}, \\ \mathcal{A}_\mu{}^{\mathcal{A}}(x) K_{\mathcal{A}} &\in \Gamma(T^*M) \otimes \mathcal{V}, \\ \mathcal{B}_{\mu\nu}{}^{\Sigma}(x) J_\Sigma &\in \Gamma(\Lambda^2 T^* X) \otimes \mathcal{B}. \end{aligned} \tag{4.11}$$

The scalars of the truncated theory arrange therefore themselves into a coset similarly to the original theory, which means that the gauge group can again be taken as a subgroup of $C_{E_{d(d)}}(G_S)$. As we saw, the Lie algebra of this commutant group is given by the singlets of $\mathrm{ad}\tilde{F}$, and because of Eq. 3.97 (and its subsequent comment), we know that T_{int} is such a singlet. This means that the singlet intrinsic torsion is a linear map:

$$T_{int} : \mathcal{V} \to \text{Lie } C_{E_{d(d)}}(G_S), \tag{4.12}$$

whose image is the gauge group. This matches the map of the embedding tensor given in Eq. 2.14. Consequently, the (opposite of the[38]) intrinsic torsion gives the embedding tensor and contains all the information for the possible gaugings of the truncated theory. In this case the quadratic constraint of Eq. 2.16 comes from taking Ξ_i in Eq. 3.97 to be another generalised vector:

$$L_{K_A} K_B = -T_{int}(K_A)_B{}^C K_C := X_{AB}{}^C K_C = \Theta_A{}^\alpha (t_\alpha)_B{}^C K_C, \tag{4.13}$$

using the equivalent definition in Eq. 2.16.

Alternatively, one can restrict the intrinsic torsion to its constant singlet part, which fixes in part the algebra (i.e. the $X_{AB}{}^C$) of Eq. 4.13. Whether generalised vectors can be constructed such that Eq. 4.13 is a different matter, which will in general restrict the algebra further.

Finally, this construction also gives the number of supercharges preserved in the truncated theory. As noted before, the number of supercharges is given by the generalised $\tilde{H}_d$ spinor bundle $\mathcal{S}$. For instance, for $\tilde{H}_d = SU(8)$, the spinor bundle is $\mathbf{8} + \bar{\mathbf{8}}$. The amount of supersymmetry preserved is consequently the number of $\tilde{G}_S$-singlets in $\mathcal{S}$.

4.4 Maximal Case

Consistent truncations retaining maximal supersymmetry[39] are possible if and only if there exists a global frame $\hat{E}_a \in \Gamma(E)$, i.e. a trivial generalised structure bundle such that:

$$[\hat{E}_a, \hat{E}_b] = X_{ab}{}^c \hat{E}_c, \tag{4.14}$$

where $X_{ab}{}^c$ are constants. If this is satisfied, the manifold is said to be "Leibniz parallelisable". This is completely analogous to the Scherk-Schwartz reduction and this construction is therefore referred as generalised Scherk-Schwartz. The reason this construction is more general than Scherk-Schwartz is that it allows most of the conventional geometry tensors in the decomposition of the generalised tangent bundle to vanish at any point as long one of them does not. For instance, consider an S^{2n} reduction of the NSNS sector of Type II supergravity described by $E \cong TM \oplus T^*M$. The hairy-ball theorem states that a global vector field would have to vanish at least at one point, meaning that the Scherk-Schwartz reduction cannot be applied. However, if at that point the covector part of E does not vanish, then a global generalised frame exists and a maximally supersymmetric consistent truncation is possible.

In fact, all spheres are Leibniz parallelisable (in the sense of generalised geometry). The condi-

[38]In order to match the convention of the embedding tensor as defined earlier, although this is not important.
[39]The emphasis will be on half and quarter maximal truncations in five dimensions rather than this case, which should be taken more as an introduction to the next sections.

tion of singlet and constant intrinsic torsion, or the generalised unimodular condition now reads:

$$X_{BA}{}^{B} = 0, \tag{4.15}$$

and the gauge fields (given in general by Eq. 4.11) take the simple form of:

$$A_\mu = A_\mu^A \hat{E}_A. \tag{4.16}$$

Defining the spheres by $\delta_{ij} y^i y^j = 1$, the global generalised frame for sphere was constructed in [15] as:

$$\hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}} A, \tag{4.17}$$

where $F = dA$ is a d-form field strength, v_{ij} are the Killing vectors on S^d and σ_{ij} are given by:

$$\sigma_{ij} = *(R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j). \tag{4.18}$$

Note that in this case a $GL^+(d+1, \mathbb{R})$ generalised geometry was used with:

$$0 \to \Lambda^{d-2} T^* M \to E \to TM \to 0, \tag{4.19}$$

similarly to the $O(d,d)$ case.

Finally the gauging will be determined by Eq. 4.14. Evaluating the Dorfman derivative on the frames of Eq. 4.17, one obtains [15]:

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = \frac{1}{R}(\delta_{ik}\hat{E}_{lj} - \delta_{il}\hat{E}_{kj} - \delta_{jk}\hat{E}_{li} + \delta_{jl}\hat{E}_{ki}). \tag{4.20}$$

So, for instance, on S^4: $X_{ij} = \frac{1}{R}\delta_{ij}$ and $X_{ijk}{}^{l} = 0$, which agrees with maximal seven-dimensional $SO(5)$ gauged supergravity.

4.5 Gauging of an $N = 4$ consistent truncation of Type IIB supergravity on SE^5

In order to show the power of this formalism, we will consider the truncation of Type IIB supergravity on a 5 dimensional Sasaki-Einstein manifold (SE_5). The first step is to calculate the intrinsic torsion in order to check if a consistent truncation exists. Since we wish to compactify Type IIB on a 5 dimensional compact manifold the generalised $SU(2)$ structure must be embedded inside $E_{5+1(5+1)}$ in some way, which we take to be:

$$E_{6(6)} \supset SO(5,5) \times SO(1,1) \supset SU(2)_S \times SO(5,2) \times SO(1,1). \tag{4.21}$$

The space of torsions [45] decomposes then as:

$$\begin{aligned}
W = \mathbf{351}' + \mathbf{27} &\xrightarrow[SO(5,5)\times SO(1,1)]{} 2.\mathbf{10_2} + \mathbf{\overline{16}_5} + \mathbf{1_8} + \mathbf{54_{-4}} + \mathbf{126_2} + \mathbf{144_{-1}} + \mathbf{1_{-4}} + \mathbf{16_{-1}} \\
&\xrightarrow[SU(2)_S\times SO(5,2)\times SO(1,1)]{} 2(\mathbf{3},\mathbf{1})_\mathbf{2} + 2(\mathbf{1},\mathbf{7})_\mathbf{2} + (\mathbf{\bar{2}},\mathbf{\bar{8}})_\mathbf{5} + (\mathbf{1},\mathbf{1})_\mathbf{8} + (\mathbf{1},\mathbf{1})_\mathbf{-4} \\
&\qquad + (\mathbf{5},\mathbf{1})_\mathbf{-4} + (\mathbf{3},\mathbf{7})_\mathbf{-4} + (\mathbf{1},\mathbf{27})_\mathbf{-4} + (\mathbf{1},\mathbf{21})_\mathbf{2} + (\mathbf{3},\mathbf{35})_\mathbf{2} \\
&\qquad + (\mathbf{2},\mathbf{8})_\mathbf{-1} + (\mathbf{4},\mathbf{8})_\mathbf{-1} + (\mathbf{2},\mathbf{48})_\mathbf{-1} + (\mathbf{1},\mathbf{1})_\mathbf{-4} + (\mathbf{2},\mathbf{8})_\mathbf{-1},
\end{aligned} \tag{4.22}$$

whereas K_G decomposes as:

$$\begin{aligned}
K_{SU(2)} = \mathbf{27} \times (\mathbf{3},\mathbf{1})_\mathbf{0} &\xrightarrow[SO(5,5)\times SO(1,1)]{} (\mathbf{1_{-4}} + \mathbf{10_2} + \mathbf{16_{-1}}) \times (\mathbf{3},\mathbf{1})_\mathbf{0} \\
\xrightarrow[SU(2)_S\times SO(5,2)\times SO(1,1)]{} &((\mathbf{1},\mathbf{1})_\mathbf{-4} + (\mathbf{3},\mathbf{1})_\mathbf{2} + (\mathbf{1},\mathbf{7})_\mathbf{2} + (\mathbf{2},\mathbf{8})_\mathbf{-1}) \times (\mathbf{3},\mathbf{1})_\mathbf{0} \\
= (\mathbf{3},\mathbf{1})_\mathbf{-4} + (\mathbf{1},\mathbf{1})_\mathbf{2} &+ (\mathbf{3},\mathbf{1})_\mathbf{2} + (\mathbf{5},\mathbf{1})_\mathbf{2} + (\mathbf{3},\mathbf{7})_\mathbf{2} + (\mathbf{2},\mathbf{8})_\mathbf{-1} + (\mathbf{4},\mathbf{8})_\mathbf{-1}
\end{aligned} \tag{4.23}$$

The singlets of the intrinsic torsion are the singlets elements of W not are contained in $K_{SU(2)}$, so:

$$W_{int} \supseteq 2(\mathbf{1},\mathbf{7})_\mathbf{2} + (\mathbf{1},\mathbf{27})_\mathbf{-4} + (\mathbf{1},\mathbf{21})_\mathbf{2} + (\mathbf{1},\mathbf{1})_\mathbf{8} + 2(\mathbf{1},\mathbf{1})_\mathbf{-4} \tag{4.24}$$

which means there is indeed a generalised singlet intrinsic torsion.[40]

The second thing that can easily be calculated is the amount of preserved supersymmetry in the truncation. As mentioned earlier, it is given by the number of G_S singlets in the decomposition of the spinor bundle, which is given by the **8** representation of $USp(8)$. The only subtlety is that the $SU(2)$ structure must be lifted as follows:

$$USp(8) \supset USp(4)_R \times USp(4) \supset USp(4)_R \times SU(2)_S \times U(1). \tag{4.25}$$

Therefore, the truncation preserves 4 singlets (i.e. half-maximal supersymmetry) as can be seen in the decomposition:

$$\mathbf{8} \to (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) \to (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2_1}) \oplus (\mathbf{1},\mathbf{2_{-1}}). \tag{4.26}$$

The third element central element of the truncation procedure is to find the generalised vectors of the consistent truncation. Their number is given by the number of G_S singlets in the decomposition of the generalised vectors (by definition in the fundamental representation of $E_{6(6)}$ is **27**) under Eq. 4.21:

$$\mathbf{27} \to \mathbf{10_2} \oplus \mathbf{16_{-1}} \oplus \mathbf{1_{-4}} \to (\mathbf{7},\mathbf{1})_\mathbf{2} \oplus (\mathbf{1},\mathbf{1})_\mathbf{-4} \oplus (\mathbf{1},\mathbf{3})_\mathbf{2} \oplus (\mathbf{8},\mathbf{2})_\mathbf{-1}, \tag{4.27}$$

which amounts to 8 singlets in the first two terms of the second decomposition such that:

$$\begin{aligned}
\mathcal{V} &= \mathbf{1_{-4}} \oplus \mathbf{7_2} \\
\{K_\mathcal{A}\} &= \{K_0, K_A\},
\end{aligned} \tag{4.28}$$

[40] Eq. 4.24 is an equality if the map $\tau : K_{SU(2)} \to W$ has no kernel, which we do not check here.

for $1 \leq A \leq 7$. In order to know the specific gauging of the truncated theory, the construction of the generalised vectors is necessary. This is done by using the equivalent of Eq. 3.64. For Type IIB $E_{6(6)}$ generalised geometry, which is given by the cubic invariant on E and E^*:

$$\begin{aligned} c(V,V,V) &= -3(\iota_v \rho \wedge \rho + \epsilon_{\alpha\beta}\rho \wedge \lambda^\alpha \wedge \lambda^\beta) - 2\epsilon_{\alpha\beta}\iota_v \lambda^\alpha \sigma^\beta, \\ c^*(Z,Z,Z) &= -3(\hat{v} \lrcorner \hat{\rho} \wedge \hat{\rho} + \epsilon^{\alpha\beta}\hat{\rho} \wedge \hat{\lambda}_\alpha \wedge \hat{\lambda}_\beta) - 2\epsilon^{\alpha\beta}\hat{v} \lrcorner \hat{\lambda}_\alpha \hat{\sigma}_\beta, \end{aligned} \tag{4.29}$$

where for Type IIB compactified on five-dimensional compact manifold, the generalised tangent bundle and its dual are given by (we take the $Gl(5,\mathbb{R})$ subgroup):

$$\begin{aligned} E &\cong TM \oplus (T^*M \oplus T^*M) \oplus \Lambda^3 T^*M \oplus (\Lambda^5 T^*M \oplus \Lambda^5 T^*M), \\ E^* &\cong T^*M \oplus (TM \oplus TM) \oplus \Lambda^3 TM \oplus (\Lambda^5 TM \oplus \Lambda^5 TM), \end{aligned} \tag{4.30}$$

such that the generalised vectors and their dual are given by[41]:

$$\begin{aligned} V &= v + \lambda^\alpha + \rho + \sigma^\alpha, \\ Z &= \hat{v} + \hat{\lambda}_\alpha + \hat{\rho} + \hat{\sigma}_\alpha. \end{aligned} \tag{4.31}$$

When constructing the generalised vectors, we require the part living in the $\mathbf{7_2}$ representation to satisfy:

$$c(K_A, K_B, K_C) = 0, \tag{4.32}$$

along with:

$$\begin{aligned} c(K_0, K_0, K_{\mathcal{A}}) &= 0, \\ c(K_0, K_{\mathcal{A}}, K_{\mathcal{B}}) = \eta_{\mathcal{AB}} &= \mathrm{diag}(-1,-1,-1,-1,-1,1,1) \\ \text{Invariance under } K_0 &\to \lambda^2 K_0, K_A \to \lambda^{-1} K_A. \end{aligned} \tag{4.33}$$

This accounts for the way $SU(2)_S$ was chosen to be embedded in Eq. 4.21, with the $SO(5,2)$ flat metric and the $U(1)$ charges specified by Eq. 4.28. Then, one wishes to construct the components of the generalised vector given by Eq. 4.31 with the conventional vectors of the Sasaki-Einstein $SU(2)$ structure, such that the algebra of Eq.4.13 closes. We will not attempt such a construction,

[41] The two indices refer to the two doublets in E and E^* following the same order in the terms in Eq. 4.30 and Eq. 4.31.

but merely show the gauging obtained given the following set of $K_{\mathcal{A}}$:

$$\begin{aligned} K_0 &= \xi, \\ K_i &= \frac{1}{\sqrt{2}}\eta \wedge j_i,\ 1 \leq i \leq 3, \\ K_4 &= \frac{1}{\sqrt{2}}(n\eta - r\mathrm{vol} - n\eta \wedge C), \\ K_5 &= \frac{1}{\sqrt{2}}(-r\eta - n\mathrm{vol} + r\eta \wedge C), \\ K_6 &= \frac{1}{\sqrt{2}}(n\eta + r\mathrm{vol} - n\eta \wedge C), \\ K_7 &= \frac{1}{\sqrt{2}}(-r\eta + n\mathrm{vol} + r\eta \wedge C), \end{aligned} \tag{4.34}$$

where n^α and r^α are contracted with the doublets of two-forms and five-forms present in Eq. 4.30, represented here respectively by two copies of η and vol respectively.

To find the gauging, the form of the generalised Lie derivative is needed and is given by (using the same notation as Eq. 4.31):

$$L_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \lambda'^\alpha - \iota_{v'}\mathrm{d}\lambda^\alpha) + (\mathcal{L}_v \rho' - \iota_{v'}\mathrm{d}\rho + \epsilon_{\alpha\beta}\mathrm{d}\lambda^\alpha \wedge \lambda'^\beta) + \mathcal{L}_v \sigma'^\alpha - \mathrm{d}\lambda^\alpha \wedge \rho' + \lambda'^\alpha \wedge \mathrm{d}\rho. \tag{4.35}$$

First, by acting with K_0:

$$\begin{aligned} L_{K_0}K_0 &= \mathcal{L}_\xi \xi = 0 \Rightarrow X_{00}{}^{\mathcal{A}} = 0, \\ L_{K_0}K_i &= \mathcal{L}_\xi(\frac{1}{\sqrt{2}}\eta \wedge j_i) \\ &= \frac{1}{\sqrt{2}}(\iota_\xi(\mathrm{d}\eta) \wedge j_i + \mathrm{d}(\iota_\xi \eta) + \eta \wedge \iota_\xi \mathrm{d}j_i + \eta \wedge \mathrm{d}(\iota_\xi j_i)) \\ &= \frac{1}{\sqrt{2}}(\iota_\xi 2j_3 \wedge j_i + \eta \wedge \iota_\xi \mathrm{d}j_i) \\ &= \begin{cases} \frac{-3}{\sqrt{2}}\eta \wedge j_2 & \text{if } j = 1, \Rightarrow X_{01}{}^2 = -3 \\ \frac{3}{\sqrt{2}}\eta \wedge j_1 & \text{if } j = 2 \Rightarrow X_{02}{}^1 = 3 \\ 0 & \text{if } j = 3 \Rightarrow X_{03}{}^{\mathcal{A}} = 0 \end{cases} \\ L_{K_0}K_{4/6} &= \frac{1}{\sqrt{2}}(\mathcal{L}_\xi \eta^\alpha + \mathcal{L}_\xi(\mp\mathrm{vol}^\alpha - (\eta \wedge C)^\alpha)) \\ &= \frac{1}{\sqrt{2}}(2\iota_\xi \mathrm{d}j_3 \mp \mathrm{d}\iota_\xi \mathrm{vol}^\alpha - \mathrm{d}(C - (\eta \wedge_\xi C))) \\ &= \frac{-1}{\sqrt{2}}\mathrm{d}C = \frac{-1}{\sqrt{2}}\kappa\mathrm{vol} = \frac{1}{\sqrt{2}}\kappa K_5 - \frac{1}{\sqrt{2}}\kappa K_7 \\ \Rightarrow X_{04}{}^5 = -X_{04}{}^7 &= \frac{1}{\sqrt{2}}\kappa = -X_{05}{}^4 = -X_{07}{}^4 \text{ by antisymmetry,} \end{aligned} \tag{4.36}$$

and in the same way, $X_{06}{}^5 = -X_{06}{}^7 = \frac{1}{\sqrt{2}}\kappa = -X_{05}{}^6 = X_{07}{}^6$.

Since $\mathrm{d}(\eta \wedge j_{1/2}) = 0$, $X_{1/2\mathcal{A}}{}^{\mathcal{B}} = 0$.

Acting with K_3, we have:

$$\begin{aligned} L_{K_3}K_0 = -2\iota_\xi(j_3\wedge j_3) = 0 \Rightarrow X_{30}{}^{\mathcal{A}} = 0, \\ L_{K_3}K_i = 0 \text{ trivially} \Rightarrow X_{3i}{}^{\mathcal{A}} = 0, 1\le i\le 3, \\ L_{K_3}\eta = \eta\wedge\frac{2}{\sqrt{2}}(j_3\wedge j_3) = \frac{4}{\sqrt{2}}\text{vol} \\ \Rightarrow X_{34}{}^5 = X_{36}{}^5 = -X_{34}{}^7 = -X_{36}{}^7 = -\sqrt{2} = X_{35}{}^4 = -X_{35}{}^6 = X_{37}{}^4 = -X_{37}{}^6. \end{aligned} \tag{4.37}$$

Finally, acting with $K_{4/5/6/7}$, we find:

$$\begin{aligned} L_\eta K_0 = -\iota_\xi \mathrm{d}\eta = 0 \Rightarrow X_{4/5/6/7\,0}{}^{\mathcal{A}} = 0, \\ L_\eta K_i = -\mathrm{d}\eta\wedge\frac{1}{\sqrt{2}}\eta\wedge j_i = -\frac{4}{\sqrt{2}}\delta_i^3\text{vol} \Rightarrow X_{4/5/6/7\,1/2}{}^{\mathcal{A}} = 0, \\ X_{43}{}^5 = -X_{47}{}^5 = -\sqrt{2} = X_{53}{}^4 = -X_{53}{}^6 = -X_{63}{}^5 = X_{63}{}^7 = X_{73}{}^4 = -X_{73}{}^6, \\ L_\eta\eta = 4j_3\wedge\eta, L_\eta\text{vol} = ... = L_\eta\eta\wedge C = 0 \\ \Rightarrow X_{45}{}^3 = X_{47}{}^3 = \sqrt{2} = -X_{54}{}^3 = -X_{74}{}^3 = X_{65}{}^3 = X_{67}{}^3 = -X_{56}{}^3 = -X_{76}{}^3. \end{aligned} \tag{4.38}$$

Note that part of the gauging could be inferred from Eq. 4.24[42] as it constrains X to be built out of (following the order the terms) the fundamental, symmetric traceless, antisymmetric and trivial representations of $SO(5,2)$, although not all necessarily contribute. As we see a more specific gauge algebra appeared, which is in fact the algebra of $\text{Heis}_3\times U(1)$, the gauge group of the truncated N=4 Type II theory on a five-dimensional Sasaki-Einstein manifold, as was already found in [53].

Similarly, the two-forms and scalars can be found, but will not be given as the only aim of this section was to show how the gauging of the truncated theory can be obtained from the formalism of [16].

4.6 Formalism of $\mathcal{N}=4$ $d=5$ Consistent Truncations of 10/11d Gauged Supergravity

As we saw, a consistent truncation of Type IIB on SE5 preserves $N=4$ supersymmetry. This is however only a particular case of such half-maximal consistent truncation. One can use generalised geometry to find all possible ways to preserve half-maximal supersymmetry, which is what will be shown here. We need to find an embedding of the double cover of the structure group G_S in $USp(8)$ such that under the decomposition induced by that embedding of the **8** representation, four G_S-singlets remain. Following the branching rules, the possible decompositions are:

$$\mathbf{8} \xrightarrow[SU(4)\times U(1)]{} \mathbf{4_1} + \mathbf{\bar{4}_{-1}}, \tag{4.39}$$

$$\mathbf{8} \xrightarrow[SU(2)\times USp(6)]{} (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{6}), \tag{4.40}$$

[42] Assuming it is an equality.

$$\mathbf{8} \xrightarrow[USp(4)_R\times USp(4)_S]{} (\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4}), \tag{4.41}$$

$$\mathbf{8} \xrightarrow[SU(2)]{} \mathbf{8}, \tag{4.42}$$

$$\mathbf{8} \xrightarrow[SU(2)\times SU(2)\times SU(2)]{} (\mathbf{2},\mathbf{2},\mathbf{2}). \tag{4.43}$$

Furthermore, the global symmetry acting on the supercharges is the R-symmetry. For four supercharges, the R-symmetry is $USp(4)$, which means that we will be interested in Eq. 4.41.[43]

Requiring exactly half-maximal supersymmetry means we have to take the double cover of G_S to be a subgroup of either group which we take to be $USp(4)_S$ such that no singlet is produced as they are already present in the other term. Turning to the bosonic sector we can investigate the possible subgroups of G_S (instead of its double cover) which is $SO(5)$. Excluding finite structure groups, the structure groups will have to be $SO(2), SO(3), SO(4)$ or $SO(5)$ which will be embedded in $E_{6(6)}$ as follows:

$$O(1,1)\times SO(5,n)\times SO(5-n)\subset SO(5,5)\times O(1,1)\subset E_{6(6)}\times R^+, \tag{4.44}$$

where $n=0,1,2,3$. Note however that subgroups of these groups can again be taken. This only concerns $SO(4)=\frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$, which produce the same singlets as $SO(4)$. However, the consistent truncation will be identical to the $SO(4)$ case as the elements of $E_{6(6)}$ and $USp(8)$ that commute with the elements of any these three subgroups are the same. Hence the scalar manifold of the truncated theory will be the same and by supersymmetry the other degrees will also be the same leading to the identical theory.

We now investigate the structure of the theory, starting with the manifold of truncated theory will be evaluated using Eq. 4.11 to:

$$\mathcal{M}_{\text{scal}} = O(1,1)\times \frac{SO(5,n)}{SO(5)\times SO(n)}, \tag{4.45}$$

where $n=0,1,2,3$. Next, the generalised vectors of the truncated theory are given by the G_S-singlets of the fundamental representation of $E_{6(6)}$. The decomposition is given by:

$$\begin{aligned}\mathbf{27} &\xrightarrow[SO(5,5)\times O(1,1)]{} \mathbf{1_{-4}}\oplus\mathbf{10_2}\oplus\mathbf{16_{-1}}\\ &\xrightarrow[SO(5)\times SO(5)\times O(1,1)]{} (\mathbf{1},\mathbf{1})_{\mathbf{-4}}\oplus(\mathbf{5},\mathbf{1})_{\mathbf{2}}\oplus(\mathbf{1},\mathbf{5})_{\mathbf{2}}\oplus(\mathbf{4},\mathbf{4})_{\mathbf{1}}.\end{aligned} \tag{4.46}$$

As stated above, this can only give a half-maximal truncation if $\mathbf{4}$ does not decompose into singlets. Therefore, one singlet comes from the first term while five come from the second one. Taking the $SO(5,n)\times SO(5-n)$ subgroup instead of $SO(5)\times SO(5)$, the singlets of $\mathbf{10_2}$ takes the form of $(\mathbf{5+n},\mathbf{1})_{\mathbf{2}}$. This matches the vector space found earlier for $n=2$ and in general we

[43]This condition is necessary as it is possible to obtain exactly four singlets by decomposing Eq. 4.39, but the supercharges would transform under a wrong R symmetry. This will still define a consistent truncation but not of supergravity.

then have:

$$\begin{aligned} \mathcal{V} &= \mathbf{1_{-4}} \oplus (\mathbf{5+n})_{\mathbf{2}}, \\ \{K_{\mathcal{A}}\} &= \{K_0, K_A : A = 1, ..., 5+n\}. \end{aligned} \tag{4.47}$$

Eq. 4.32 and 4.33 remain unchanged except that metric η is now replaced by the $SO(5,n)$ flat metric.

Generalised two-forms live in the $N \cong \det T^*M \otimes E^*$ generalised bundle, whose representation has the same number of dimensions as E. The $SO(5,5)$ representations are then the same as before except $\mathbf{16}$ is replaced by $\mathbf{16'}$, which does not contribute to any singlets. The space $\mathcal{B}$ of generalised two-forms has therefore the same form as the space of generalised vectors:

$$\begin{aligned} \mathcal{B} &= \mathbf{1_4} \oplus (\mathbf{5+n})_{\mathbf{-2}}, \\ \{J^{\mathcal{A}}\} &= \{J^0, J^A : A = 1, ..., 5+n\}, \\ \langle J^{\mathcal{A}}, K_{\mathcal{B}} \rangle &= \delta^{\mathcal{A}}_{\mathcal{B}} = \text{vol}. \end{aligned} \tag{4.48}$$

The last equation only specifies a normalisation of the two-forms, which is possible because of the isomorphism $N \cong \det T^*M \otimes E^*$, the volume form itself being isomorphic to the determinant part. The generalised two-forms are therefore completely determined by the generalised vectors, which can be seen even more explicitly using the cubic invariant. The first decomposition of Eq. 4.46 is equivalent to:

$$E \cong E_0 \oplus E_{10} \oplus E_{16}, \tag{4.49}$$

which means that the truncated metric can simply be seen as the sum of three separate metrics in the sense that:

$$G = G_0 + G_{10} + G_{16}. \tag{4.50}$$

The ten-dimensional metric follows exactly the construction we gave earlier for the $O(d,d)$ case. First we split into negative-definite and positive-definite eigenspaces (i.e $E_{10} = C_- \oplus C_+$ meaning $G_{10} = G_+ + G_-$) under the maximal compact subgroup $SO(5) \times SO(5)$ such that the difference of the metrics of each eigenspace gives the $SO(5,5)$ invariant metric (i.e. $\eta(V,V) = G_+ - G_-$). Secondly we find the metric on one of these eigenspaces. This can be done because we already have a basis for C_- given by the $SO(5)$ part of Eq. 4.33, leading to:

$$G_-(V,V) = -\eta^{\mathcal{AB}}|_{SO(5)} \langle K^*_{\mathcal{A}}|_{SO(5)}, V \rangle \langle K^*_{\mathcal{B}}|_{SO(5)}, V \rangle = \delta^{ab} \langle K^*_a, V \rangle \langle K^*_b, V \rangle \,, \tag{4.51}$$

where we renamed the indices to label only the $SO(5)$ part (i.e. the first five generalised vectors in the $\mathbf{5+n}$ representation). We then have:

$$\begin{aligned} G_{10}(V,V) &= 2G_-(V,V) + \eta(V,V), \\ &= 2\delta^{ab} \langle K^*_a, V \rangle \langle K^*_b, V \rangle + \eta(V,V). \end{aligned} \tag{4.52}$$

G_0 denotes the singlet part and is given straightforwardly by:

$$G_0(V,V) = \langle K_0^*, V\rangle \langle K_0^*, V\rangle \, . \tag{4.53}$$

G_{16} is found using the Mukai pairing which we did not present earlier. We therefore only give the result:

$$G_{16} = -4\sqrt{2}\, \langle K_1 ... K_5 . V, V\rangle \, . \tag{4.54}$$

The point is that, like G_0 and G_{10}, G_{16} can also be given explicitly only in terms of the generalised vectors. Similarly, as we saw, the gauging also only depends on the generalised vectors. This means that a half-maximal consistent truncation is entirely specified once the generalised vectors are found. In fact, the generalised vectors specify completely the embedding of the structure group inside $E_{6(6)}$, which therefore specifies the whole truncation.

5 $\mathcal{N} = 2$ $d = 5$ consistent Truncations of 10/11-dimensional Gauged Supergravity

5.1 Determination of the Possible Structure Groups G_S

We know wish to apply the same formalism to the case of quarter maximal ($\mathcal{N} = 2$) supersymmetry truncations (following closely [17]). One reason to consider this theory is that two complex supercharges is the smallest non-zero number of supercharges that can be considered. This can be understood as follows. As shown in [28], when compactifying eleven-dimensional supergravity on a six-dimensional manifold, Cliff(10,1; $\mathbb{R}$) decomposes as:

$$(\mathbf{10}, \mathbf{1}) \to (\mathbf{4}, \mathbf{1}) + (\mathbf{6}, \mathbf{0}). \tag{5.1}$$

Since Cliff(4,1; $\mathbb{R}$) is isomorphic to the matrix group $M(2, \mathbb{H})$, where $\mathbb{H}$ denotes the quaternions, which excludes $\mathcal{N} = 1$ in five dimensions.

The starting point is identical: enumerating the possible decompositions of the spinor bundle that retain two singlets under G_S. Since we are in the same number of dimensions, we can find the necessary information from Eq. 4.39 - 4.43. Combined with the fact that the R symmetry for two supercharges is $SU(2)$, it is clear that the only way to produce only two singlets is to use the $SU(2) \times USp(6) \subset USp(8)$ embedding, taking $USp(6)$ as the double cover of the structure group. One can decompose $\mathbf{6}$ further as long as no new singlet is produced. These further possible decompositions are:

$$\begin{aligned}(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}) &\xrightarrow[SU(2)\times SU(3)\times U(1)]{} (\mathbf{2}, \mathbf{1})_{\mathbf{0}} + (\mathbf{1}, \mathbf{3})_{\mathbf{1}} + (\mathbf{1}, \bar{\mathbf{3}})_{\mathbf{-1}} \\ &\xrightarrow[SU(2)^2\times U(1)^2]{} (\mathbf{2}, \mathbf{1})_{\mathbf{0},\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{\mathbf{-2},\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{\mathbf{1},\mathbf{1}} + (\mathbf{1}, \mathbf{1})_{\mathbf{2},\mathbf{-1}} + (\mathbf{1}, \mathbf{2})_{\mathbf{-1},\mathbf{-1}}.\end{aligned} \tag{5.2}$$

From the first embedding, the structure group could be: $SU(3)$, $U(1)$ or $SU(3) \times U(1)$; from the second, it could be: $SU(2) \times U(1)$, $U(1)$ (not the same as the first embedding $U(1)$), $U(1)^2$ or $SU(2) \times U(1)^2$.

$$\begin{aligned}&(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{6}) \xrightarrow[SU(2)^2\times USp(4)]{} (\mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{4}) \\ &\begin{cases} \xrightarrow[SU(2)^4]{} (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \\ \qquad \xrightarrow[SU(2)^2\times SU(2)]{} (\mathbf{2}, \mathbf{1}, \mathbf{1}) + 2\cdot(\mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}) \\ \qquad \xrightarrow[SU(2)^2\times U(1)]{} (\mathbf{2}, \mathbf{1})_{\mathbf{0}} + 2\cdot(\mathbf{1}, \mathbf{1})_{\mathbf{1}} + 2\cdot(\mathbf{1}, \mathbf{1})_{\mathbf{-1}} + (\mathbf{1}, \mathbf{2})_{\mathbf{0}} \\ \\ \xrightarrow[SU(2)^3\times U(1)]{} (\mathbf{2}, \mathbf{1}, \mathbf{1})_{\mathbf{0}} + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{\mathbf{1}} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{\mathbf{-1}} \\ \xrightarrow[SU(2)^3]{} (\mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{4}) \end{cases}\end{aligned} \tag{5.3}$$

From the first embedding, G_S (or its double cover) could be: $SU(2) \times USp(4)$; from the second embedding: three of the $SU(2)^3 \subset SU(2)^4$, one of $SU(2)^2 \subset SU(2)^3 \times U(1)$ or $SU(2)^2 \times U(1) \subset SU(2)^3 \times U(1)$ or one of $SU(2)^2 \subset SU(2)$; from the last two embeddings: two of $SU(2)^2 \subset SU(2)^2 \times SU(2)$ and both $SU(2) \times U(1) \subset SU(2)^2 \times U(1)$.

$$\begin{aligned} (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{6}) &\xrightarrow[SU(2)^2]{} (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{6}) \\ &\xrightarrow[SU(2)\times U(1)]{} \mathbf{2_0} + \mathbf{1_5} + \mathbf{1_3} + \mathbf{1_1} + \mathbf{1_{-1}} + \mathbf{1_{-3}} + \mathbf{1_{-5}} \end{aligned} \tag{5.4}$$

Hence, $G_S = SU(2)$ from the first embedding or $G_S = U(1)$ from the second one.

$$\begin{aligned} (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{6}) &\xrightarrow[SU(2)^3]{} (\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{2}) \\ &\begin{cases} \xrightarrow[SU(2)^2\times U(1)]{} (\mathbf{2},\mathbf{1})_\mathbf{0} + (\mathbf{1},\mathbf{3})_\mathbf{1} + (\mathbf{1},\mathbf{3})_\mathbf{-1} \\ \xrightarrow[SU(2)^2\times U(1)]{} (\mathbf{2},\mathbf{1})_\mathbf{0} + (\mathbf{1},\mathbf{2})_\mathbf{2} + (\mathbf{1},\mathbf{2})_\mathbf{0} + (\mathbf{1},\mathbf{2})_\mathbf{-2} \end{cases} \end{aligned} \tag{5.5}$$

So, from the first embedding one of two $G_S = SU(2)$ or $G_S = SU(2)^2$ is possible and from the last two: $G_S = U(1)$ only in the penultimate line or $G_S = SU(2) \times U(1)$ can be realised.

These are all the possible continuous structure groups giving a consistent $\mathcal{N} = 2$ truncation. However, apart from the fact that the differential condition imposed by the intrinsic torsion was not looked at yet, these embeddings could give rise to the same truncation if the content of the theory is the same. This is the reason why the decompositon of Eq. 5.4 can be ignored.

5.2 Determination of the Hypermultiplet Moduli Space

Having found the possible structure groups, we now turn to the structure of the truncated theory. Given one of the structure groups specified above, Eq. 4.11 can be used to find the moduli of scalars. This can be further restricted by the fact that for $N = 2$, $d = 5$ supergravity, the scalar manifold takes the form:

$$\mathcal{M}_{\text{scal}} = \mathcal{M}_{\text{VT}} \times \mathcal{M}_{\text{H}}, \tag{5.6}$$

where V stands for vectors, T for tensors and H for hypermultiplets. This can be understood as follows. We start by taking the largest G_S possible, which from the last section is $USp(6)$. The bosonic embedding strucure has to be modified from Eq. 4.40 to the central product:

$$E_{6(6)} \supset USp(6) \cdot SU(2) = \frac{USp(6) \times SU(2)}{\mathbb{Z}_2}. \tag{5.7}$$

In this way, we have that the generalised $E_{6(6)}$ vectors and adjoint tensors decompose as:

$$\begin{aligned} \mathbf{27} &= (\mathbf{1},\mathbf{1}) \oplus (\mathbf{14},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2}) \\ \mathbf{78} &= (\mathbf{1},\mathbf{3}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{21},\mathbf{1}) \oplus (\mathbf{14},\mathbf{1}) \oplus (\mathbf{14'},\mathbf{2}), \end{aligned} \tag{5.8}$$

which can be obtained by first decomposing into $SU(6) \times SU(2)$ and then taking $USp(6) \times SU(2)$.

We see then that a generalised $USp(6)$-structure will be equivalently defined by one generalised vector and three generalised adjoint tensors (the four $USp(6)$ singlets). The **3** representation of $SU(2)$ means that the triplet of J_α satisfy (with a choice of normalisation):

$$[J_\alpha, J_\beta] = 2\epsilon_{\alpha\beta\gamma} J_\gamma, \quad \mathrm{tr}(J_\alpha J_\beta) = -\delta_{\alpha\beta}. \tag{5.9}$$

This is called an "H structure". There is only one invariant generalised vector K whose norm is positive:

$$c(K, K, K) := 6\kappa^2, \tag{5.10}$$

where κ is defined in this way to be a section of $(\det T^*M)^{1/2}$. This is called a "V structure". In total, the four generalised singlets further must satisfy a compatibility condition:

$$J_\alpha \cdot K = 0, \tag{5.11}$$

using the adjoint action described for eleven-dimensional supergravity in the Appendix E of [39], which defines what is called an "HV structure". The compatibility condition forces the deformations of K that leave the invariant J_α and inversely, which means that the scalar manifold takes the product structure of Eq. 5.6.

More generally, G_S will be a smaller group that does not produce more singlets in the spinor bundle decomposition. The generalised tangent bundle will decompose then as:

$$E_{6(6)} \supset G_S \cdot C_{E_{6(6)}}(G_S), \tag{5.12}$$

and G_S could produce more singlets in $\mathbf{27}^*$ and $\mathbf{78}$. Depending on the representations appearing in the decompositions of either, the generalised invariant vectors K_I and adjoint tensors J_A will obey different algebras than that presented above. For instance the J_A will form a group O:

$$[J_A, J_B] = f_{AB}{}^{C} J_C, \tag{5.13}$$

where $f_{AB}{}^{C}$ are the structure constant of $\mathfrak{o}$. In fact, one can go further using the fact that J_A are singlets in the adjoint of $E_{6(6)}$ under G_S, which means that $O \subset C_{E_6(6)}(G_S)$[44].

However, the compatibility condition will remain true for all invariant J_A and K_I:

$$J_A \cdot K_I = 0. \tag{5.14}$$

In fact, Eq. 5.14 can be used to find the stabiliser of the V structure, which enables us to find the commutant in Eq. 5.12. This can be done because the structure group will be forced to be a subgroup of the stabiliser group, thereby specifying the needed embedding.

The names of the V and H structures come from the type of multiplet of supergravity. Five-

[44] This is a similar reasoning to the one used in Eq. 4.9.

dimensional $\mathcal{N}=2$ gauged supergravity has four types of multiplets:

$$\begin{aligned} 1 \text{ Gravity multiplet: } & \{g_{\mu\nu}, \psi^i_\mu, A_\mu\}, \\ n_{\mathrm{V}} \text{ Vector multiplets: } & \{A_\mu, \lambda^i, \phi\}, \\ n_{\mathrm{T}} \text{ Tensor multiplets: } & \{B_{\mu\nu}, \lambda^i, \phi\}, \\ n_{\mathrm{H}} \text{ Hypermultiplets: } & \{\zeta^i, q^u\}, \end{aligned} \tag{5.15}$$

where $i = 1, 2$ (since $\mathcal{N}=2$) and $u = 1, ..., 4$. There are then $\mathrm{n_{V+1}}$ vector fields A_μ, 4 $\mathrm{n_H}$ scalars from the hypermultiplets that parameterise $\mathcal{M}_H$, a quaternionic Kähler manifold and $\mathrm{n_{V+T}}$ scalars from the tensor and vector multiplets that parameterise $\mathcal{M}_{VT}$, which is called a very special real manifold. Finally, using Eq. 4.11 and Eq. 5.6, the scalar manifold of the truncated theory will separate as:

$$\mathcal{M}_{\mathrm{scal}} = \frac{C_{G_U}(G_S)}{C_{H_U}(G_S)} \times \frac{C_{G_V}(G_S)}{C_{H_V}(G_S)} =: \frac{G_{\mathrm{VT}}}{H_{\mathrm{VT}}} \times \frac{G_{\mathrm{H}}}{H_{\mathrm{H}}}, \tag{5.16}$$

where G_U and G_V are subgroups of $E_{6(6)}$ while H_U and H_V are subgroups of $\frac{USp(8)}{\mathbb{Z}_2}$. Note that the groups defined on the right-hand side of Eq. 5.16 correspond to the groups remaining once the possible common factors have been cancelled (except in the case of no hypermultiplet, where we take $G_H = H_H = SU(2)$). As mentioned, $O \subset C_{E_6(6)}(G_S)$ and because the compatibility condition of Eq. 5.14, $C_{E_6(6)}(G_S)$ splits as in Eq. 5.16. This means that we can identify O and G_H and the hypermultiplet manifold will be given by:

$$M_H = \frac{G_H}{SU(2) \cdot C_{USp(6)}(G_S)}. \tag{5.17}$$

The number of vector multiplets will then be given by the dimension of $\mathcal{V} - 1$, while the singlets in the adjoint bundle contribute to the truncated scalar manifold too.

To summarise, requiring exactly two supersymmetries in the five dimensional truncated supergravity enables to list all possible structure groups as done in Eq. 4.39-4.43. From this, the truncated scalar manifold, as the number of generalised vectors and sections of the generalised adjoint bundle can be found. Using the structure of $N = 2$ supergravity in five dimensions, we see that the algebra of the J_A's is contained in part of the scalar manifold. As before, the representation of the generalised G_S-singlet intrinsic torsion gives the most general possible gauging[45]. In total, going through all possible G_S listed above, only 9 theories without hypermultiplet (including up to 14 vector and tensor multiplets maximum), 5 theories with 1 hypermultiplet (including up to 4 vector and tensor multiplets maximum) and 1 theory with 2 hypermultiplets (no vector or tensor multiplet allowed) are possible consistent truncations of ten- or eleven- dimensional supergravity.[46] This is more restrictive than what was thought to be possible. For instance, when there is no hypermultiplet, it was known that the the very special manifold could take the form

[45]To be more specific, the bracket defined by the generalised Lie derivative definse a Leibniz algebra (as it is not antisymmetric but respects the Leibniz product rule). From this, a double-sided ideal can be constructed. The quotient of $\mathcal{V}$ by this ideal can be used to find the gauge algebra (i.e. not the local symmmetry, just the group under which the matter content is charged as defined in Section 2.3.)

[46]The table of possible truncations is given in [17].

of $M_{VT} = \mathbb{R}^+ \times \frac{SO(n_{VT}-1,1)}{SO(n_{VT}-1)}$, for arbitrary n_{VT} or also $M_{VT} = \frac{E_{(6,-26)}}{F_4}$ for $n_{VT} = 26$. However, because we know from Eq. 5.8 that $n_{VT} \leq 14$, these manifolds are excluded for arbitrary vector and tensor multiplets.

Note that the approach has been entirely algebraic so far, no differential constraint has been solved yet (i.e. checking if the K_I's and J_A's can be constructed on a manifold admitting a singlet intrinsic torsion and satisfying the right algebras). We then expect that this will further restrict the possible truncations to a smaller number than the 20 cases presented above. This second part must be done individually for each possible theory, and in the next section, we shall show an attempt at explicitely constructing the sections of the generalised adjoint bundle for one of the possibilities in Type IIB.

5.3 Non existence of a Specific Geometric Consistent Truncation

We now investigate the case of $n_H \geq 1, n_{VT} \geq 1$. In order to find the structure group (and its embedding) corresponding to a chosen number of multiplets with at least one hypermultiplet, it is easier to start with the symmetric hypermultiplet manifold, given in Eq. 5.17. The compatibility condition implies that the stabiliser of the generalised vector is $F_{4(4)}$, so that $G_S \subset F_{4(4)} \subset E_{6(6)}$, which constrains Eq. 5.17 to four possible families [54] [17], two of which preserve only two singlets in the decomposition of **8**:

$$\begin{aligned} M_H &= \frac{G_{2(2)}}{SO(4)},\ n_H = 2, \\ M_H &= \frac{SU(2,1)}{S(U(2)\times U(1))},\ n_H = 1. \end{aligned} \tag{5.18}$$

G_2 is fourteen-dimensional, which means there are 14 J_A's in the first manifold whereas there are 8 J_A's in the second one, corresponding respectively to $G_S = SU(2)$ and $G_S = SU(3)$. Explicitely, the decompositions are in the first case (corresponding to the spinor decomposition of Eq. 5.5) :

$$\begin{aligned} \mathbf{27} &\xrightarrow[F_{4(4)}]{} \mathbf{1} + \mathbf{26} \xrightarrow[SU(2)\times G_{2(2)}]{} (\mathbf{1},\mathbf{1}) + (\mathbf{5},\mathbf{1}) + (\mathbf{3},\mathbf{7}) \\ \mathbf{78} &\xrightarrow[F_{4(4)}]{} \mathbf{26} + \mathbf{52} \xrightarrow[SU(2)\times G_{2(2)}]{} (\mathbf{3},\mathbf{1}) + (\mathbf{5},\mathbf{1}) + (\mathbf{3},\mathbf{7}) + (\mathbf{5},\mathbf{7}) + (\mathbf{1},\mathbf{14}) \end{aligned} \tag{5.19}$$

and in the second case (corresponding to the spinor decomposition of Eq. 5.2):

$$\begin{aligned} \mathbf{27} &\xrightarrow[F_{4(4)}]{} \mathbf{1} + \mathbf{26} \xrightarrow[SU(3)\times SU(2,1)]{} (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{8},\mathbf{1}) \\ \mathbf{78} &\xrightarrow[F_{4(4)}]{} \mathbf{26} + \mathbf{52} \xrightarrow[SU(3)\times SU(2,1)]{} (\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}}) + 2(\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{8}) + (\bar{\mathbf{6}},\mathbf{3}) + (\mathbf{6},\bar{\mathbf{3}}). \end{aligned} \tag{5.20}$$

In order to have a non-zero number of vector and tensor multiplets, we need to find the subgroups of $SU(2)$ and $SU(3)$ that produce more singlets in the decomposition of **27**. In the first case breaking $SU(2)$ further to some $U(1)$ implies the presence of additional singlets in the spinor bundle (as seen by the earlier decomposition of the spinor bundle). This means that if $n_H = 2$, then a consistent truncation can only have $n_{VT} = 0$.

Turning to the second case, we find by decomposing Eq. 5.20 to $SU(3) \supset SU(2) \times U(1)$ that:

$$\begin{aligned} &(\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{8},\mathbf{1}) \xrightarrow[SU(2)\times SU(2,1)\times U(1)]{} \\ &2(\mathbf{1},\mathbf{1})_{\mathbf{0}} + (\mathbf{2},\mathbf{1})_{\mathbf{3}} + (\mathbf{2},\mathbf{1})_{\mathbf{-3}} + (\mathbf{1},\mathbf{3})_{\mathbf{-2}} + (\mathbf{1},\bar{\mathbf{3}})_{\mathbf{2}} + (\mathbf{3},\mathbf{1})_{\mathbf{0}} + (\mathbf{2},\mathbf{3})_{\mathbf{1}} + (\mathbf{2},\bar{\mathbf{3}})_{\mathbf{-1}}, \end{aligned} \tag{5.21}$$

which has five $U(1)$-singlets and hence $n_{VT} = 4$. Note that this creates more adjoint singlets too:

$$\begin{aligned} &(\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}}) + 2(\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{8}) + (\bar{\mathbf{6}},\mathbf{3}) + (\mathbf{6},\bar{\mathbf{3}}) \xrightarrow[SU(2)\times SU(2,1)\times U(1)]{} \\ &2(\mathbf{1},\mathbf{1})_{\mathbf{0}} + 2(\mathbf{2},\mathbf{1})_{\mathbf{3}} + 2(\mathbf{2},\mathbf{1})_{\mathbf{-3}} + (\mathbf{1},\mathbf{3})_{\mathbf{4}} + (\mathbf{1},\mathbf{3})_{\mathbf{-2}} + \\ &(\mathbf{1},\bar{\mathbf{3}})_{\mathbf{2}} + (\mathbf{1},\bar{\mathbf{3}})_{\mathbf{-4}} + 2(\mathbf{3},\mathbf{1})_{\mathbf{0}} + 2(\mathbf{2},\mathbf{3})_{\mathbf{1}} + 2(\mathbf{2},\bar{\mathbf{3}})_{\mathbf{-1}} + (\mathbf{3},\mathbf{3})_{\mathbf{-2}} + (\mathbf{3},\bar{\mathbf{3}})_{\mathbf{2}} + (\mathbf{1},\mathbf{8})_{\mathbf{0}}, \end{aligned} \tag{5.22}$$

eight of which will contribute to the scalar manifold.

We proceed to the construction of the generalised adjoint tensors that parametrise the second option of Eq. 5.18, i.e. we start by looking for the structure constants of $\mathfrak{su}(3)$. First, we need the generalised adjoint bundle which for $E_{6(6)}$, $d = 5$, is given by:

$$\mathrm{ad}\tilde{F} \cong \mathbb{R} \oplus (S \otimes S^*)_0 \oplus (TM \otimes T^*M) \oplus (S \otimes \Lambda^2 TM) \oplus (S \otimes \Lambda^2 T^*M) \oplus \Lambda^4 TM \oplus \Lambda^4 T^*M, \tag{5.23}$$

where $l = \frac{r^a_a}{3}$ (we restrict ourselves to $E_{d(d)} \subset E_{d(d)} \times R^+$) such that:

$$R = l + a_i + r + \beta^i + B^i + \gamma + C, \tag{5.24}$$

where $l \in \mathbb{R}$, $a_i \in (S \otimes S^*)_0$ with $i \in \{1, 2, 3\}$, $r \in TM \otimes T^*M$,...

In he half-maximal case, $T^{1,1}$ admitted a consistent truncation with eight generalised vectors and adjoint tensors. In that case, the J_A's can then be found from the generalised vectors, but not here. One way to construct the consistent truncation is then by starting with invariant tensors, which is what we do here.

It is possible although not necessary for the quarter maximal consistent truncation with three generalised vectors and eight generalised adjoint tensors to intersect the previous case. It is conjectured here that $T^{1,1}$ admits this quarter maximal consistent truncation, sharing the same minimum of the potential as the previous truncation on $AdS \times T^{1,1}$, possessing one generalised vector and three generalised adjoint tensors. The generalised vector should belong to $\mathcal{V}_{1/2}$ as well as $\mathcal{V}_{1/4}$ and is given by:

$$K = \xi - \eta \wedge j_3. \tag{5.25}$$

Since only this vector is related to the previous half-maximal consistent truncation, we make the stronger conjecture that the generalised adjoint bundle parametrising $\mathfrak{su}(3)$ can be built from only one of the three two-forms that enter the $T^{1,1}$ structure, in addition to the Reeb vector and η - that is, the content that enters Eq. 5.25.[47]

[47]This conjecture was made by Prof. Waldram, who also suggested to look at this stronger version in order to tackle the problem more easily.

This particular consistent truncation is an interesting example as it would potentially leads to a further intrinsically generalised geometric consistent truncation (apart from the maximally supersymmetric spheres truncations).

We aim to construct 8 independent sections of 5.23 from the Reeb vector ξ, the 1-form η, and the 2-form j_3. These obey the following relations (the full relations obeyed by the tensors in our choice of basis of a Sasaki-Einstein manifold can be found in Section 4.2.1 of [16]):

$$\begin{aligned} \xi \lrcorner j_3 &= 0 \\ \xi \lrcorner \eta &= 1 \\ \frac{1}{2} j_3 \wedge j_3 \wedge \eta &= vol_5 \\ d\eta &= 2j_3 \\ dj_3 &= 0 \end{aligned} \tag{5.26}$$

We take the following specific realisation from Eq.4.60 of [16]:

$$\begin{aligned} \xi &= -3\partial_\psi \\ \eta &= -e^1 \\ j_3 &= e^{25} - e^{34} \end{aligned} \tag{5.27}$$

We can then construct R with $B^i = j_3^i$, $\beta^i = j_3^{*i}$, $C = vol_4 = -j_3 \wedge j_3 = 2e^2 \wedge e^3 \wedge e^4 \wedge e^{548}$, $\gamma = vol_4^* = -j_3^* \wedge j_3^* = 2\hat{e}_2 \wedge \hat{e}_3 \wedge \hat{e}_4 \wedge \hat{e}_5$, $a_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $r = -2\hat{e}_1 \otimes e^1 + 2\hat{e}_2 \otimes e^2 + 2\hat{e}_3 \otimes e^3 + 2\hat{e}_4 \otimes e^4 + 2\hat{e}_5 \otimes e^5$, and $l = \frac{r^a{}_a}{3} = 2$.

Using Eq. E.35 of [39], calculating the commutator $[R, R']$, where $R, R' \in \mathrm{ad}\tilde{F}$ amounts to the following 14 components commutators:

$$\begin{aligned} [\gamma, C'] &= [vol_4^*, vol_4] = l_1'' + r_1'' \\ l_1'' &= \frac{1}{2} vol_4^* \lrcorner vol_4 = \frac{1}{2} 2.2 = 2 = l \\ r_1'' &= (jvol_4^* \lrcorner jvol_4) - \frac{1}{2}\mathbb{I}(vol_4^* \lrcorner vol_4) \\ &= -2\hat{e}_1 \otimes e^1 + 2\hat{e}_2 \otimes e^2 + 2\hat{e}_3 \otimes e^3 + 2\hat{e}_4 \otimes e^4 + 2\hat{e}_5 \otimes e^5 = r \end{aligned} \tag{5.28}$$

using that:

$$(jvol_4^* \lrcorner jvol_4)^a{}_b = \frac{1}{3!}(vol_4^*)^{ac_1c_2c_3}(vol_4)_{bc_1c_2c_3} = \frac{1}{3!}.3!.2.2\delta_b^{a\neq 1} = 4\delta_b^{a\neq 1} \tag{5.29}$$

[48] By j_3^*, we mean that j_{3ij}'s indices were raised with the metric. Then, $j_3^* = \hat{e}_{25} - \hat{e}_{34}$, where $e^i(\hat{e}_j) = \delta^i_j$, and we have $j_3^* \lrcorner j_3 = 2$.

$$
\begin{gathered}
[\beta^k, B'^l] = [j_3^{*k}, j_3^l] = l_2'' + r_2'' + a_2'' \\
l_2'' = \frac{1}{4}\epsilon_{kl} j_3^{*k} \lrcorner j_3^l = \begin{cases} \frac{1}{2} = \frac{l}{4} & \text{if } k=1, l=2 \\ -\frac{1}{2} = -\frac{l}{4} & \text{if } k=2, l=1 \\ 0 & \text{else.} \end{cases} \\
r_2'' = \epsilon_{kl}(j j_3^{*k} \lrcorner j j_3^l) - \frac{1}{4}\mathbb{I}\epsilon_{kl} j_3^{*k} \lrcorner j_3^l \\
= \begin{cases} -\frac{1}{2}\hat{e}_1 \otimes e^1 + \frac{1}{2}\hat{e}_2 \otimes e^2 + \frac{1}{2}\hat{e}_3 \otimes e^3 + \frac{1}{2}\hat{e}_4 \otimes e^4 + \frac{1}{2}\hat{e}_5 \otimes e^5 = \frac{r}{4} & \text{if } k=1, l=2 \\ +\frac{1}{2}\hat{e}_1 \otimes e^1 - \frac{1}{2}\hat{e}_2 \otimes e^2 - \frac{1}{2}\hat{e}_3 \otimes e^3 + \frac{1}{2}\hat{e}_4 \otimes e^4 - \frac{1}{2}\hat{e}_5 \otimes e^5 = -\frac{r}{4} & \text{if } k=2, l=1 \\ 0 & \text{else,} \end{cases}
\end{gathered} \tag{5.30}
$$

using that:

$$
(j j_3^* \lrcorner j j_3)^a_{\ b} = \frac{1}{(2-1)!} j_3^{*ac} j_{3bc} = \begin{cases} 1 & \text{if } a = b \text{ and } a \neq 1 \\ 0 & \text{else.} \end{cases} \tag{5.31}
$$

Since $r_2'' \propto r_1''$, we can take $r = r_2''$

$$
\begin{gathered}
(a_2'')^i_{\ j} = \epsilon_{jk} j_3^{*i} \lrcorner j_3^k - \frac{1}{2}\delta^i_j \epsilon_{kl} j_3^{*k} \lrcorner j_3^l \\
= \begin{cases} \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = a_1 & \text{if } k=1, l=1 \\ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = a_2 & \text{if } k=2, l=2 \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a_3 & \text{if } k=1, l=2 \\ \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a_3 & \text{if } k=2, l=1 \end{cases}
\end{gathered} \tag{5.32}
$$

$$
\begin{gathered}
[r, r'] = r_3'' \\
r_3''{}^a_{\ b} = (r.r')^a_{\ b} = r^a_{\ c}\alpha r^c_{\ b} - r^c_{\ b}\alpha r^a_{\ c} = 0,
\end{gathered} \tag{5.33}
$$

taking $r' = \alpha r$.

$$
\begin{gathered}
[a_i, a_j'] = a_4'' = a.a \\
a_4''{}^a_{\ b} = (a_i.a_j')^a_{\ b} = a_i{}^a_{\ c} a_j{}^c_{\ b} - a_i{}^c_{\ b} a_j{}^a_{\ c} = \begin{cases} -4a_3 & \text{if } i=1, j=2 \\ -2a_1 & \text{if } i=1, j=3 \\ 2a_3 & \text{if } i=2, j=3 \\ 0 & \text{if } i=j, \end{cases}
\end{gathered} \tag{5.34}
$$

where a_4'' is antisymmetric in i, j.

$$[r, \beta'^i] = \beta_5^i = 4j_3^*$$
$$\beta_5^{i\,ab} = (r.j_3^{*i})^{ab} = r^a_{\ c} j_3^{*icb} + r^b_{\ c} j_3^{*iac} \tag{5.35}$$

$$[a_x, \beta'^i] = [a_x, j_3^{*i}] = \beta_6^i$$
$$\beta_6 = a_x.j_3^* = \begin{cases} -2j_3^{*1} & \text{if } x = 1, i = 2 \\ 2j_3^{*2} & \text{if } x = 2, i = 1 \\ j_3^{*1} & \text{if } x = 3, i = 1 \\ -j_3^{*2} & \text{if } x = 3, i = 2 \\ 0 & \text{else.} \end{cases} \tag{5.36}$$

$$[\gamma, B'^i] = [vol_4^*, j_3^i] = \beta_7^i$$
$$\beta_7^i = -vol_4^* \lrcorner j_3'^i = 2(\hat{e}_3 \wedge \hat{e}_4 - \hat{e}_2 \wedge \hat{e}_5) = -2j_3^* \tag{5.37}$$

$$[r, B'^i] = B_8^i = -4j_3^i$$
$$B_8^i = r.j_3'^i \tag{5.38}$$

$$[a_x, B'^i] = B_9^i$$
$$B_9^i = a_x.j_3 = \begin{cases} 2j_3^1 & \text{if } x = 1, i = 2 \\ -2j_3^2 & \text{if } x = 2, i = 1 \\ -j_3^1 & \text{if } x = 3, i = 1 \\ j_3^2 & \text{if } x = 3, i = 2 \\ 0 & \text{else.} \end{cases} \tag{5.39}$$

$$[\beta^i, C'] = [j_3^{*i}, vol_4] = B_{10}^i$$
$$B_{10}^i = j_3^{*i} \lrcorner vol_4' = -2j_3 \tag{5.40}$$

$$[r, \gamma'] = \gamma_{11}$$
$$\gamma_{11} = r.vol_4^* = 8vol_4^* \tag{5.41}$$

$$[\beta^i, \beta'^j] = [j_3^{*i}, j_3^{*j}] = \gamma_{12}$$
$$\gamma_{12} = \epsilon_{ij} j_3^{*i} \wedge j_3^{*j} = \begin{cases} -vol_4^* & \text{if } i = 1, j = 2 \\ vol_4^* & \text{if } i = 2, j = 1 \\ 0 & \text{else.} \end{cases} \tag{5.42}$$

$$[r, C'] = [r, vol_4] = C_{13}$$
$$C_{13} = r.vol_4 = -8vol_4 \tag{5.43}$$

$$[B^i, B'^j] = [j_3^i, j_3^j] = C_{14}$$
$$C_{14} = -\epsilon_{ij} j_3^i \wedge j_3^j = \begin{cases} -vol_4 & \text{if } i = 1, j = 2 \\ vol_4 & \text{if } i = 2, j = 1 \\ 0 & \text{else.} \end{cases} \tag{5.44}$$

Recapitulating, taking $J_1 = a_1, J_2 = a_2, J_3 = a_3, J'_4 = r, J_5 = j_3^{*1}, J_6 = j_3^{*2}, J_7 = j_3^1, J_8 = j_3^2, J_9 = vol_4^*, J_{10} = vol_4, J_{11} = 2$, as well as eliminating J_{11} by the rescaling $J_4 = J'_4 + J_{11}$, we obtain the following non-zero commutators (the rest being deduced from their antisymmetry):

$$
\begin{aligned}
[J_1, J_3] = -2J_1,\ [J_2, J_3] = 2J_3,\\
[J_1, J_2] = -4J_3,\\
\\
[J_4, J_5] = 4J_5,\ [J_4, J_6] = 4J_6,\\
[J_4, J_7] = -4J_7,\ [J_4, J_8] = -4J_8,\\
[J_4, J_9] = 8J_9,\ [J_4, J_{10}] = -8J_{10},\\
\\
[J_5, J_6] = -J_9,\ [J_7, J_8] = -J_{10},\\
[J_9, J_7] = -2J_5,\ [J_9, J_8] = -2J_6,\\
[J_{10}, J_5] = 2J_7,\ [J_{10}, J_6] = 2J_8,\\
[J_9, J_{10}] = J_4,\\
\\
[J_1, J_6] = -2J_5,\ [J_1, J_8] = 2J_7,\\
[J_2, J_5] = 2J_6,\ [J_2, J_7] = -2J_8,\\
[J_3, J_5] = J_5,\ [J_3, J_6] = -J_6,\\
[J_3, J_7] = -J_7,\ [J_3, J_8] = J_8,\\
[J_5, J_7] = J_1,\ [J_5, J_8] = \frac{J_4}{4} + J_3,\\
[J_6, J_7] = -\frac{J_4}{4} + J_3,\ [J_6, J_8] = J_2.
\end{aligned}
\tag{5.45}
$$

Taking:

$$
\begin{aligned}
J'_A = i\frac{J_1 + J_2}{2}, J'_B = i\frac{J_1 - J_2}{2}, J'_C = J_3,\\
J_D = \frac{J_9 + J_{10}}{2}, J_E = \frac{J_9 - J_{10}}{2}, J_F = \frac{J_4}{4},
\end{aligned}
\tag{5.46}
$$

we have two $su(2)$ algebras as shown below:

$$\begin{aligned}
[J'_A, J'_B] = -2J'_C,\ [J'_A, J'_C] = -2J'_B,\ [J'_B, J'_C] = -2J'_A, \\
[J_D, J_E] = -2J_F,\ [J_D, J_F] = -2J_E,\ [J_E, J_F] = -2J_D, \\
[J'_{A/B/C}, J_{D/E/F}] = 0,
\end{aligned}$$

$$\begin{aligned}
[J'_A, J_5] = iJ_6,\ [J'_B, J_5] = -iJ_6,\ [J'_C, J_5] = J_5, \\
[J'_A, J_6] = -iJ_5,\ [J'_B, J_6] = -iJ_5,\ [J'_C, J_6] = -J_6, \\
[J'_A, J_7] = -iJ_8,\ [J'_B, J_7] = iJ_8,\ [J'_C, J_7] = -J_7, \\
[J'_A, J_8] = iJ_7,\ [J'_B, J_8] = iJ_7,\ [J'_C, J_8] = J_8,
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
[J_D, J_5] = J_7,\ [J_E, J_5] = -J_7,\ [J_F, J_5] = J_5, \\
[J_D, J_6] = J_8,\ [J_E, J_6] = -J_8,\ [J_F, J_6] = J_6, \\
[J_D, J_7] = -J_5,\ [J_E, J_7] = -J_5,\ [J_F, J_7] = -J_7, \\
[J_D, J_8] = -J_6,\ [J_E, J_8] = -J_6,\ [J_F, J_8] = -J_8,
\end{aligned}$$

$$\begin{aligned}
[J_5, J_6] = -J_D - J_E,\ [J_5, J_7] = -i(J'_A + J'_B),\ [J_5, J_8] = J'_C + J_F, \\
[J_7, J_8] = J_D - J_E,\ [J_6, J_8] = -i(J'_A - J'_B),\ [J_6, J_7] = J'_C - J_F.
\end{aligned}$$

Since the two $su(2)$ algebras commute with each other and there is no $su(2) \times su(2)$ in $su(3)$, the only possibility of finding an $su(3)$ subalgebra in this 10-dimensional algebra is by setting to 0 two generators of either $su(2)$ algebra. However, because of the last two lines, this cannot be done. Hence, there is no $su(3)$ subalgebra here.[49]

In fact, one could go further and try to identify this algebra. One ansatz can be constructed by noting that $so(4) \cong su(2) \times su(2)$, we can embed $so(4)$ in $so(5)$, implying the following structure:

$$\begin{pmatrix} u_{ij} & v_i \\ -v_j & 0 \end{pmatrix}, [u, u'] \in so(4), [v, v'] \in so(4), [u, v] \sim v' \tag{5.48}$$

which resembles the algebra we have, where $u_{ij} \in so(4)$ and $1 \leq i, j \leq 4$.

[49]As an unrelated aside, we see also that $\frac{G}{SU(2) \times SU(2)}$, where G is the 10-dimensional group in question, is a symmetric space, as we have the structure $[h, h] \subset h, [h, f] \subset f, [f, f] \subset h$, where $h = su(2)$. [55]

6 Conclusion

This dissertation was devoted to understanding how to construct consistent truncations of supergravity using generalised geometry, beyond the case of Leibniz parallelisable spaces. The two most central elements that needed to be understood for it to work were generalised G-structures and generalised intrinsic torsion.

We started by quickly reviewing the concept of G-structures and saw how familiar geometrical concepts could be recast in this language. We followed by defining some terms relevant in the context of gauged supergravity, which gave at the same time a context to study generalised geometry.

This introduction being over, generalised geometry was constructed starting from the actions of Type II and eleven-dimensional supergravities, with the main of arriving at the definition of the generalised intrinsic torsion. The main steps of the construction were as follows.

First, the symmetries of these actions, including the fluxes, were discussed. This is similar to Maxwell's theory except there is more than one flux and they are not necessarily two-forms. The generators of these symmetries are consequently patched in a non-trivial way. The space in which these generators live as well as their patching are the two data necessary to define the generalised tangent bundle, which unifies all the symmetries (diffeomorphism and gauge). This is done through the concept of exact sequences which was introduced beforehand. Because these were exact sequences of vector spaces, we could use a theorem that implies that the generalised tangent bundle is in fact isomorphic to the naive definition that would have been used without knowing about the patching.

However, understanding it was required to define the generalised Lie derivative as well as the bracket, which formed the second step of the overall construction of the theory. The fact that the Lie derivative can be generalised is non-trivial and points already to a new language for supergravity. This was dependent on two things: first (and most importantly) the patching induced by the action, secondly the right choice of integration. The properties of these brackets can be used to define generalised geometry in terms of more abstract structures called algebroids (specific algebraic construction with in some sense enough structure to capture the geometry of supergravity) which was not covered here.

The third step was to unify all the bosonic fields in the generalised metric. This was an important early instance of G-structure in the context of generalised geometry, where the analogy with the corresponding notion in conventional geometry is the most clearly seen. This agreed with the remark made when introducing gauged supergravity that scalars arrange themselves in cosets.

These first steps were done side by side for both the NSNS sector of Type II supergravity and the low-energy limit of M theory to emphasize the unity of the idea that animates complex generalised geometry and exceptional generalised geometry as well as singling out some of the subtleties that arise in practice in the latter case. For example, the exceptional case must be done once the theory is already compactified. This in turn is why the patching rule for the dualised metric was not a direct consequence of supergravity, but was only found retroactively by the action of $E_{7(7)} \times R^{+}$.

Being equipped with a metric, generalised connections compatible both with the $O(d,d)\times R^+$ or the $E_{d(d)}\times R^+$ structure and the generalised metric could be defined, which are important types of connections, similarly to the conventional case. This lead to the definition of the generalised torsion, based on the Dorfman derivative and a metric-compatible generalised connection. The same relation holds in conventional geometry, although only as a property in this case (not as the definition).

An important aspect by which generalised geometry departs from the common intuition is that the fundamental theorem of Riemannian geometry does not hold: generalised torsion-free, generalised metric-compatible connections are not unique, although only unique projections appear in supergravity. Defining a map τ from the space of differences of compatible connections to the space of torsions, the intrinsic torsion is the torsion modulo the image of that map. Under some conditions, the intrinsic torsion can be found by simply decomposing the space of torsion and comparing it to the decomposition of domain of τ.

The intrinsic torsion is important for two main reasons. First, it is an intrinsic measure of the geometry considered (through the G-structure) since it is by definition the part of the torsion that is independant of the choice of connection. The second reason is that it controls whether the derivative of singlets remain singlets.

Finally, we mentioned how the bosonic part of the supergravity actions could be rewritten as a generalised Einstein-Hilbert action, which is the natural culmination of the successful unification of the symmetries of ten- and eleven-dimensional supergravities in a single generalised diffeomorphism.

The second part was devoted to the construction of consistent truncations, following the theorem of [16]. Some more motivation to study consistent truncations of supergravity was given, focusing on its place in the context of the AdS/CFT correspondence, where the lack of separation of scales represents an obstacle. Following Scherk and Schwartz, the primary condition to the existence of a truncation not giving a merely effective description of the theory by turning off the heavy modes was to keep only and all the singlet modes in the decomposition of the fields. When formulated in conventional geometry, the Scherk-Schwartz truncation scheme could account for parallelisable spaces, which was already an important step beyond the Kaluza-Klein original circular example.

Since the geometry of supergravity was shown to be generalised geometry, their idea could be enlarged to Leibniz parallelisable spaces. In this context, a global generalised frame for spheres was given, thereby showing that all spheres are Leibniz parallelisable spaces. This gave a deeper reason for the existence of truncations (known to be consistent) on non-parallelisable (in the conventional sense) spheres. This again was abstracted to generalised G-structures. Using this and the intrinsic torsion, the structure of the theory can be inferred. Given a singlet intrinsic torsion, the manifold of scalars is given by the embedding of the G-structure inside $E_{d(d)}\times R^+$ through the commutant, while the representations that appear in the singlet intrinsic torsion determine the most general gauging.

In order to show how this truncation scheme works, two cases were investigated. First, an

$\mathcal{N} = 4$ consistent truncation of Type IIB supergravity on SE_5 was checked to be consistent. This was done by explicitly evaluating the Dorfman derivative on given generalised vectors, which gave a gauging by $Heis_3 \times U(1)$. The more general case was then reviewed by showing how to obtain the possible structure groups, and hence the spaces of generalised vectors. These in turn determined the generalised adjoint tensors as well the three parts of the generalised metric.

Since there is less supersymmetry, the case of $\mathcal{N} = 2$ is more difficult. We started by again giving the possible (non-discrete) structure groups by decomposing the spinor bundle, which was again reduced as two distinct G-structures can sometimes lead to the same truncated theory. Next we turned our attention to the scalar manifold. Using the same formula as before, it was seen that it splits into a tensor part and a hypermultiplet part, which is consistent with quarter-maximal five-dimensional supergravity.

From this algebraic analysis, it seems possible to retain a hypermultiplet along with some tensor multiplets. In the low-energy of M-theory, this already was shown to be realised. It was then conjectured that it happened in Type IIB as well. A strong form of this conjecture was shown to not be possible. This in not so surprising as a number of assumptions were made when we tried to construct all the generalised tensors by keeping only three invariant Saski-Einstein tensors. Nevertheless, this illustrated how a possible ansatz could be constructed using generalised geometry. At this point then, further work is required to know whether such a consistent truncation exists.

7 Appendix

7.1 Notation

The wedge symbol is used in the same way for both forms and antisymmetric tensors $w \in \Lambda^p TM$. The j-notation is defined in components by [28; 36]:

$$
\begin{aligned}
(jw \wedge w')^{m,m_1\ldots m_7} &:= \frac{7!}{(p-1)!(8-p)!} w^{m[m_1\ldots m_{p-1}} w'^{m_p\ldots m_7]}, \\
(j\lambda \wedge \lambda')_{m,m_1\ldots m_7} &:= \frac{7!}{(q-1)!(8-q)!} \lambda_{m[m_1\ldots m_{q-1}} \lambda'_{m_q\ldots m_7]}, \\
(jw \lrcorner j\lambda)^m{}_n &:= \frac{1}{(p-1)!} w^{mn_1\ldots n_{p-1}} \lambda_{nn_1\ldots n_{p-1}}, \\
(jt \lrcorner j\tau)^m{}_n &:= \frac{1}{7!} t^{m,n_1\ldots n_7} \tau_{n,n_1\ldots n_7}.
\end{aligned}
\tag{7.1}
$$

Antisymmetric vector-like tensors are used throughout, following the same component conventions as forms, so that for a p-polyvector w and a $q-$form λ, the contractions and wedge products are given by:

$$
\begin{aligned}
(w \wedge w')^{m_1\ldots m_{p+p'}} &= \frac{(p+p')!}{p!p'!} w^{[m_1\ldots m_p} u^{m_{p+1}\ldots m_{p+p'}]}, \\
(\lambda \wedge \lambda')_{m_1\ldots m_{q+q'}} &= \frac{(q+q')!}{q!q'!} w_{[m_1\ldots m_q} u_{m_{q+1}\ldots m_{q+q'}]}, \\
(w \lrcorner \lambda)_{a_1\ldots a_{q-p}} &:= \frac{1}{p!} w^{n_1\ldots n_p} \lambda_{n_1\ldots n_p a_1\ldots a_{q-p}}, \quad \text{if } q \geq p, \\
(w \lrcorner \lambda)^{a_1\ldots a_{p-q}} &:= \frac{1}{q!} w^{a_1\ldots a_{p-q} n_1\ldots n_q} \lambda_{n_1\ldots n_q}, \quad \text{if } p \geq q.
\end{aligned}
\tag{7.2}
$$

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